

(Noelia)

GALOIS ACTION ON  
THE PRINCIPAL BLOCK

[j.w M. Schaeffer Fry & C. Vallejo]

I - Introduction

$G$  finite group,  $p$  prime  $p \mid |G|$

$P \in \text{Syl}_p(G)$ ,  $|G|_p = p^a = |P|$

$\text{Irr } G$  irred. characters (complex valued)

$X(G)$ : character table of  $G$

Problem 12 [Brauer] Does  $X(G)$  determine  
if  $P$  is abelian?

A) 1995 [Kimmeli - Sawding]

$X(G) = X(H) \quad Q \in \text{Syl}_p(G)$

then  $P$  abelian  $\Leftrightarrow Q$  abelian and in that case  $P \trianglelefteq Q$

Problem 23 (BHZ) Brauer's height zero conjecture

$\text{Irr } B = \text{Irr}_0 B \Leftrightarrow D$  is abelian

- ( $\Leftarrow$ ) Kessar-Malle '13
- ( $\Rightarrow$ ) Redouad, Navarro-Späth '14

$B \in Bl(G)$  p-block of  $G$ ,  $\text{Irr } G = \bigsqcup_{B \in Bl(G)} \text{Irr } B$

$B \rightsquigarrow \{D^g\}$   $D$  defect group, p-subgroup of  $G$

$|D| = p^{d(B)}$   $d(B)$  = defect of  $B$

Then  $\min \{ \chi(1)_p \mid \chi \in \text{Irr } B \} = p^{a-d(B)}$

$\rightsquigarrow \forall \psi \in \text{Irr } B, \psi(1)_p = p^{a-d(B)+h}$   
where  $h \geq 0$  is the height of  $\psi$

$\text{Irr}_0(B) = \{ \psi \in \text{Irr } B \mid h=0 \}$  height zero chars.

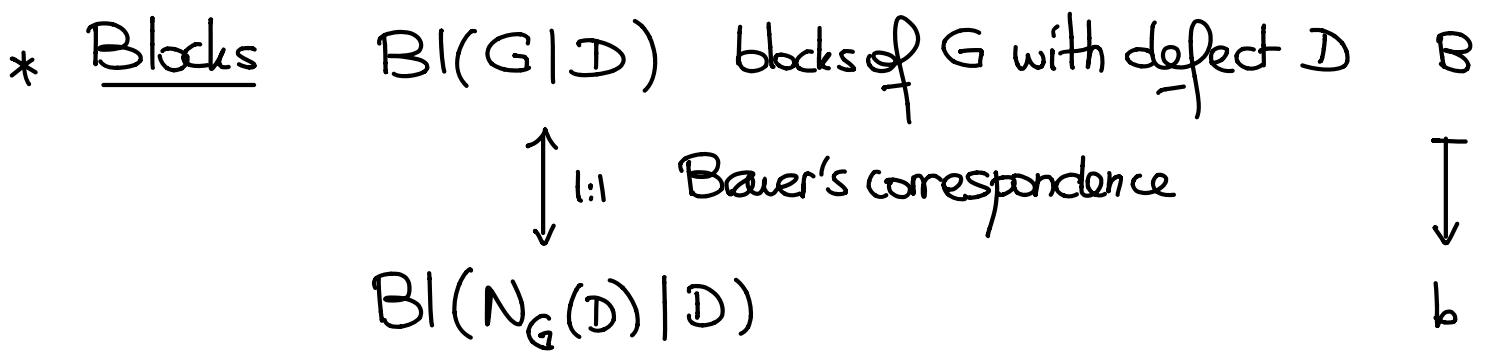
$B_0(G)$  principal block (s.t  $1_G \in \text{Irr } B_0(G)$ )

$$1 = 1_G(1)_p = p^{a-d(B_0)+h} \Rightarrow a = d(B_0) \\ \Rightarrow D \in \text{Syl}_p(G)$$

## 2- Global-local conjecture

Conj [McKay]  $|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(N_G(P))|$

- Reduced '07 by Isaacs-Malle-Navarro
- '16 Malle-Späth for  $p=2$



Conj [Alperin-Mckay]     $|\text{Irr}_B| = |\text{Irr}_b|$

- reduced, Späth '13  
 $\rightsquigarrow$  inductive A.M condition

\* Galois action     $|G| = n$ ,  $\zeta$  primitive  $n$ th root of 1

$X \in \text{Irr } G$  then  $\forall g \in G \quad X(g) \in \mathbb{Q}(\zeta)$

$\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) =: \mathcal{G}$  acts on  $\text{Irr}(G)$  by

$$X^\sigma(g) := (X(g))^\sigma$$

$\mathcal{H} := \{\sigma \in \mathcal{G} \mid \exists r \geq 0 \quad \forall \eta \text{ p' root of 1} \text{ then } \sigma(\eta) = \eta^{p^r}\}$

Galois-Mckay Conj [Navarro]    for all  $\sigma \in \mathcal{H}$

$\boxed{|\text{Irr}_p(G)^\sigma| = |\text{Irr}_p(N_G(P))^\sigma|}$

- $p$ -solvable [Turull]

- cyclic Sylow [Navarro]
- alternating groups [Brunat - Nath]
- Lie type in defining char. [Ruhstorfer]
- reduced [Navarro - Späth - Vallejo]

\* Blocks and Galois

Blockwise Galois-McKay [Navarro '04]  $\sigma \in \mathcal{H}$

$$\boxed{|(\text{Irr}_o B)^\sigma| = |(\text{Irr}_o b)^\sigma|}$$

- $p$ -solvable [Turull]
- cyclic defect [Navarro]

3 - A particular element of  $\mathcal{H}$

$e \geq 1$   $\sigma_e$  fixes  $p$ ' roots of 1  
 sends  $\eta$   $p$ -power root of 1 to  $\eta^{p^e+1}$   
 $\Rightarrow \sigma_e \in \mathcal{H}$

Thm [Navarro-Tiep]  $\text{Irr}_{p'}(B_o(G)) = \text{Irr}_{p'}(B_o(G))^{\sigma_e}$   
 $\Rightarrow \exp(P/[P,P]) \leq p^e$

$$\underline{\text{Conj}} \quad [N-T] \quad \text{Irr}_{p'}(G) = \text{Irr}_{p'}(G)^{\sigma_p} \Leftrightarrow \exp(P/[P,P]) \leq p^e$$

- Malle  $p=2$

#### 4- Our work

Thm A  $p=2, 3$

$$\boxed{|\text{Irr}_{p'}(B_o(G))^{\sigma_1}| = p \Leftrightarrow \text{P cyclic}}$$

- $p > 3$   $G = D_{2p}$  ( $\Leftarrow$ ) doesn't hold
- $p = 5, 7, 11$  ( $\Rightarrow$ ) consequence of blockwise Galois-McKay

Lemma  $\text{Lin}(P)^{\sigma_1} = \text{Irr}(P) \Phi(P)$

$\uparrow$  Frattini group

proof : " $\leq$ "  $\lambda \in \text{Lin}(P)^{\sigma_1}, \lambda \neq 1_P$

$$\lambda(x)^{\sigma_1} = \lambda(x)^p \lambda(x)$$

"

$$\lambda(x) \text{ hence } \lambda^p = 1$$

$$\text{i.e. } [P : \ker \lambda] = o(\lambda) = p$$

$$\Rightarrow P/\ker \lambda \cong \mathbb{Z}/p\mathbb{Z} \text{ and therefore } \ker \lambda \geq \Phi(P)$$

" $\geq$ " :  $\lambda(x)^{\sigma_1} = \lambda(x)^p \lambda(x)$

$$= \lambda(x \Phi(P))^p \lambda(x \Phi(P)) = \lambda(x) \quad \square$$

- \*  $P$  cyclic iff  $|P/\Phi(P)| = p$
- iff  $|(\text{In } P)^{\sigma_1}| = p$
- iff  $|\text{In}_{\bar{P}}(B_o(P))^{\sigma_1}| = p$

Thm [Navarro, RSV] If  $P \trianglelefteq G$

$$\text{In}_{\bar{P}}(B_o(G))^{\sigma_1} = \text{In}(G/\Phi(P)O_{\bar{P}}(G))$$

[Fong]  $G$   $p$ -solvable  $\Rightarrow \text{In}(B_o(G)) = \text{In}(G/O_p(G))$

$$\bar{G} = G/\Phi(P)O_{\bar{P}}(G), \quad \bar{P} = P/\Phi(P)$$

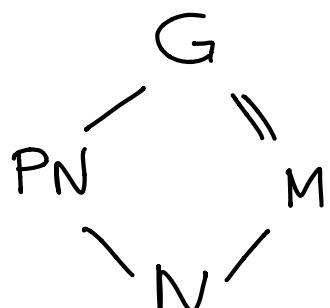
$$\text{if } \text{In}(\bar{G}) = p \in \{2, 3\}$$

$$\begin{cases} p=2 \\ \bar{G} = C_2 = \bar{P} \end{cases}$$

$$\begin{cases} p=3 \\ \bar{G} = C_3 = \bar{P} \\ \text{or } \bar{G} = \zeta_3 \quad \bar{P} = C_3 \end{cases}$$

\* CFSG

$$\bullet p=2$$



Cyclic 2-Sylow  
 $\Rightarrow \exists$  normal complement

- ( $p=3$ ) [Herzog - Brauer]  
 and  $P$  cyclic

$\begin{cases} p\text{-solvable} \\ \text{if } M \trianglelefteq G \text{ then} \\ p \nmid |M| \text{ or } p \nmid [G:M] \end{cases}$

Thm B  $S$  non-abelian simple

- $p=2 \quad 1_S, \varphi_1, \varphi_2 \in \text{Irr}_2'(B_o(S))^{G_i}$   
 $\varphi_1, \varphi_2$  are not  $\text{Aut}(S)$ -conjugate  
 $X/S \in \text{Syl}_2(\text{Aut}(S)/S) \quad \varphi_i$  is  $X$ -invariant
- $p=3$ 
  - \*  $P$  cyclic  $| \text{Irr}_3'(B_o(S))^{G_i} | = 3$   
 $1_S \neq \varphi_1, \varphi_2 \in \text{Irr}_3'(B_o(S))^{G_i}$  not  $\text{Aut}(S)$ -conj.  
and  $\varphi_i$   $X$ -invariant
  - \*  $P$  non-cyclic  $1_S, \varphi_1, \varphi_2, \varphi_3 \in \text{Irr}_3'(B_o(S))^{G_i}$   
 $\varphi_1, \varphi_2, \varphi_3$  not  $\text{Aut}(S)$ -conj. and  $\varphi_i$   $X$ -invariant

Conj :  $p=2, 3$

$$| \text{Irr}_o(B)^{G_i} | = p \iff D \text{ is cyclic}$$