(Lucas)

JORDAN DECOMPOSITION FOR THE ALPERIN-MCKAY CONJECTURE

I – Local global conjectures

- G finite group, ℓ prime $\ell \mid |G|$
- $(K, \mathcal{O}, k)$ ℓ-modular system, large enough

$\rightsquigarrow$ $\mathcal{O}G = \bigoplus_{i=1}^{r} B_i$ dec. into ℓ-blocks

$\rightsquigarrow$ $\operatorname{Irr}_K(G) = \operatorname{Irr} G = \coprod_{i=1}^{r} \operatorname{Irr} B_i$

Brauer : to an ℓ-block B associates

* a defect group D, ℓ-group, unique up to conj.
* an ℓ-block b of $N_G(D)$ of defect D

Conjecture [Broué] if D is abelian then

$$D^b(B\text{-mod}) \simeq D^b(b\text{-mod})$$

Conjecture [Alperin-Mckay] There exists a bijection

$$\operatorname{Irr}_0 B \xleftrightarrow{1:1} \operatorname{Irr}_0 b$$

Recall $\mathrm{Irr}_0 B = \{ \chi \in \mathrm{Irr}\, B \mid \chi(1)_\ell = [G:D]_\ell \}$

<u>Thm</u> [Späth '13] The AM-conjecture holds for all finite simple groups if the inductive AM-conjecture (iAM) holds for all finite simple groups

Let S be a finite non-abelian simple group and G its universal covering group
Then (iAM) holds for a block B of G if there exists an $\mathrm{Aut}(G)_{B,D}$-equivariant bijection

$$\Omega : \mathrm{Irr}_0 B \xrightarrow{\sim} \mathrm{Irr}_0 b$$

"preserving the Clifford theory" wrt $G \trianglelefteq G \rtimes \mathrm{Aut}(G)_{B,D}$

- extendibility of characters on both sides is preserved:
  $\chi \in \mathrm{Irr}_0(B)$ extends to $G \rtimes X$, $X \leq \mathrm{Aut}(G)_\chi$
  $\iff \Omega(\chi)$ ——————— $N_G(D) \rtimes X$
- more generally: cocycle in $H^2(X/G, K^\times)$ given by χ on $\Omega(\chi)$ coincide

<u>difficulty</u>: need to explicitly construct projective representations.

II - Representation theory of groups of Lie type

From now on : G connected reductive / $\overline{\mathbb{F}}_p$ $p \neq \ell$
(e.g. $Gl_n(\overline{\mathbb{F}}_p)$, ...)
$F : G \longrightarrow G$ Frobenius endomorphism

Aim : understand rep. theory of G^F

[Broué-Michel '89] Let (G^*, F^*) be in duality with (G, F). Then

$$\mathcal{O}G^F = \bigoplus_{(s)} \mathcal{O}G^F e_s^{G^F}$$

where (s) runs over G^{*F^*}-conjugacy classes of semisimple elements of G^{*F^*} of ℓ'-order

Aim : understand the representations of $\mathcal{O}G^F e_s^{G^F}$ for a fixed (s).

Deligne-Lusztig varieties : P parabolic $P = L \ltimes U$ with $F(L) = L$

$$Y_U^G := \{ gU \mid g^{-1}F(g) \in U.F(U) \} \subseteq G/U$$

G^F (resp. L^F) acts by left (resp. right) multiplication

$\rightsquigarrow H^i_c(Y^G_U, \mathcal{O})$ is an $\mathcal{O}G^F$-$\mathcal{O}L^F$-bimodule

Let L^* be the minimal F-stable Levi-subgroup such that $L^* \supseteq C_{G^*}(s)$

$\rightsquigarrow L$ in duality with L^*

Thm [Bonnafé–Rouquier '03] The bimodule

$$H_c^{\dim Y^G_U}(Y^G_U, \mathcal{O}) e_s^{L^F}$$

induces a Morita equivalence b/w $\mathcal{O}G^F e_s^{G^F}$ and $\mathcal{O}L^F e_s^{L^F}$

def: a block B of $\mathcal{O}G^F e_s^{G^F}$ is called quasi-isolated if $C_{G^*}(s)$ is not contained in a proper Levi of G^*

Thm yields a reduction to quasi-isolated blocks.

III - Applications to the (iAM)-condition

From now on: G simple, simply-connected s.t $G^F/Z(G^F)$ is a non-abelian simple gp with universal covering group G^F

(e.g $G = SL_n$ for n, q not too small.)

Lemma: There exists a "nice" subgroup $E \leq \mathrm{Aut}(G^F)$ s.t. the image of $\langle \mathrm{Diag}(G^F), E \rangle$ in $\mathrm{Out}(G^F)$ is $\mathrm{Out}(G^F)_{L, e_s^{G^F}}$.

Thm A: if $\ell \gg 0$ then

$$\mathcal{O}L^F E e_s^{L^F}\text{-mod} \simeq \mathcal{O}G^F E e_s^{G^F}\text{-mod}$$

Thm B: Assume that every quasi-isolated block of a finite quasi-simple group of Lie type satisfies (iAM) then if G^F is not of type D the (iAM) holds for $S = G^F/Z(G^F)$.

"only need to check (iAM) for quasi-isolated blocks".