ENDOTRIVIAL MODULES FOR THE SPORADIC SIMPLE GROUPS AND THEIR COVERS

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ABSTRACT. In a step towards the classification of endotrivial modules for quasi-simple groups, we investigate endotrivial modules for the sporadic simple groups and their covers. A main outcome of our study is the existence of torsion endotrivial modules with dimension greater than one for several sporadic groups with p-rank greater than one.

1. INTRODUCTION

Let G be a finite group and k a field of prime characteristic p dividing the order of G. A kG-module V is called endotrivial if $V \otimes V^* \cong k \oplus Q$, with Q a projective kG-module. The tensor product over k induces a group structure on the set of isomorphism classes of indecomposable endotrivial kG-modules, called the group of endotrivial modules and denoted $T(G)$. This group is finitely generated and it is of particular interest in modular representation theory as it forms an important part of the Picard group of self-equivalences of the stable category of finitely generated kG-modules. In particular the self-equivalences of Morita type are induced by tensoring with endotrivial modules.

Endotrivial modules have seen a considerable interest since defined by Dade in 1978 in [Dad78a, Dad78b] where he gives a classification for abelian p-groups. In this case $T(G)$ is cyclic and generated by the module $\Omega(k)$. Since then a full classification has been obtained over p-groups (see [Car12] and the references therein). Contributions towards a general classification of endotrivial modules have already been obtained for several families of general finite groups ([CMN06, CMN09, CHM10, MT07, CMT11, CMT13, LMS13, NR12, Rob11]), however the problem of computing $T(G)$ for an arbitrary finite group G remains open in general. In particular the problem of determining the structure of the torsion subgroup $TT(G)$ of $T(G)$ is the hardest part. In [Car12, Thm. A] J. Carlson states that in the known cases, but for few exceptions, $TT(G) = X(G)$, the group of one-dimensional kG-modules.

In this article, we study endotrivial modules for the sporadic simple groups and their covering groups, i.e. perfect groups G such that $G/Z(G)$ is sporadic simple, in all characteristics p dividing the order of G. As a consequence we obtain a significant list of new examples where $TT(G)$ is non-trivial, and a fortiori not isomorphic to $X(G)$.

Our first main result is the determination of the torsion-free rank of $T(G)$ for all covering groups G of sporadic groups (see Proposition 4.1 and Section 5). Our second main result is the full structure of $T(G)$ for all covering groups of sporadic groups in characteristic 2 (see

Section 4). Our third main result concerns proper covering groups, i.e. cases where $Z(G)$ is non-trivial: in any such case, we either prove that $T(G) \cong T(G/Z(G))$ via inflation, or compute $T(G)$ by explicit arguments (see Section 4 and Section 6). Our fourth main result is the full structure of the group $T(G)$ in all characteristics where the Sylow p-subgroups are cyclic (see Table 5), although in this case the main theoretical ingredients are provided by [MT07].

At this point we are left with the computation of the torsion subgroup $TT(G)$ in odd characteristic p such that p^2 divides the order of G. In this paper we determine the full structure of $TT(G)$ in all characteristics for the following groups: M_{11}, M_{12}, J_1, M_{22}, J_2, M_{23}, ${}^2F_4(2)'$, HS, M_{24}, McL, Suz, Fi_{22}, HN, Ly, J_4 and their covers. In the remaining cases, we do not obtain the full structure of $TT(G)$ for all $p \,|\, |G|$, but we give as much information as possible. Typically, in most missing cases we can provide an upper bound in terms of abelian groups for $TT(G)$.

Our methods combine on the one hand traditional modular representation theory techniques such as the theory of vertices and sources (see [Car12]) and on the other hand character theory. More specifically we use the well-known fact that trivial source modules lift uniquely to characteristic zero and use the newly obtained characterisation of the torsion subgroup $TT(G)$ via ordinary characters of the group G (see Theorem 2.11). In particular this approach allows us to compute Green correspondence at the level of ordinary characters, rather than at the level of modules, allowing us to make computer calculations via the GAP Character Table Libraries even for large sporadic groups. Whenever possible we provide the characters afforded by trivial source endotrivial module, so that using the known decomposition matrices, one can recover the composition factors of these modules. In addition, we rely on a recently developed approach by Balmer ([Bal13]) which uses weak H-homomorphisms. Leaving aside the details, Balmer's method is useful in helping to find the trivial source endotrivial modules, though it does not provide a complete answer in general.

Our paper is organised as follows. In Sections 2 and 3, we sum up useful results on endotrivial modules and the sporadic groups. In Section 4, we determine the structure of $T(G)$ in characteristic 2. For odd characteristics, in Section 5 we determine the torsion-free rank of $T(G)$, and in Section 6 the torsion subgroup $TT(G)$ when we have obtained its full structure. We collect the results in Section 7 in Table 3 if $p^2 \,|\, |G|$ and in Table 5 if the Sylow p-subgroups are cyclic.

2. Endotrivial modules: background results

Throughout, unless otherwise stated, we let G be a finite group, P a Sylow p-subgroup of G, for some prime p dividing the order of G, and k an algebraically closed field of characteristic p. We write $N_G(P)$ for the normaliser of P in G and we adopt the same notation and conventions as in [Thé95]. In particular, all kG-modules are left modules and finitely generated. In addition, for a kG-module M, we put $M^* = \text{Hom}_k(M, k)$ for the k-dual of M and $\otimes = \otimes_k$. Also, $X(G)$ denotes the group of isomorphism classes of 1-dimensional kG-modules.

2.1. **Definitions and elementary properties.**

Definition 2.1. A kG-module M is *endotrivial* if $\text{End}_k(M) \cong k \oplus (\text{proj})$ as kG-modules, where (proj) denotes some projective kG-module.

Here are the main basic properties of endotrivial modules.

Lemma 2.2. *Let M be a kG-module.*

(1) *M is endotrivial if and only if M splits as the direct sum $M_0 \oplus (\text{proj})$ for an indecomposable endotrivial kG-module M_0, unique up to isomorphism.*

(2) *The relation*
$$M \sim N \iff M_0 \cong N_0$$
on the class of endotrivial kG-modules is an equivalence relation and every equivalence class contains a unique indecomposable module up to isomorphism.

(3) *If M is endotrivial, then $M{\downarrow}_H^G$ is endotrivial for any subgroup H of G. Moreover if $H \geq P \in \text{Syl}_p(G)$, then M is endotrivial if and only if $M{\downarrow}_H^G$ is.*

(4) *M is endotrivial if and only if M^* is endotrivial.*

(5) *Given two endotrivial kG-modules M, N, then $M \otimes N$ is endotrivial.*

(6) *Let $0 \longrightarrow M \longrightarrow Q \longrightarrow N \longrightarrow 0$ be a short exact sequence of kG-modules with Q projective. Then M is endotrivial if and only if N is endotrivial.*

(7) *If M is endotrivial, then*
$$\dim_k(M) \equiv \begin{cases} \pm 1 \pmod{|G|_p} & \text{if } p \text{ is odd} \\ \pm 1 \pmod{\frac{1}{2}|G|_2} & \text{if } p = 2 \end{cases}$$
where $|G|_p$ is the order of a Sylow p-subgroup of G. Also, if M is moreover a trivial source module, then $\dim_k(M) \equiv 1 \pmod{|G|_p}$.

For proofs of Lemma 2.2(1)-(7), we refer the reader to [Car12] and the references therein. Part (2) is equivalent to saying that two endotrivial modules M, N are equivalent if and only is they are isomorphic in the stable module category of kG. This leads us to the following definition:

Definition 2.3. The set $T(G)$ of equivalence classes of endotrivial kG-modules resulting from the relation $\sim$ of Lemma 2.2(2) forms an abelian group for the composition law
$$[M] + [N] := [M \otimes N].$$
The zero element of $T(G)$ is the class $[k]$ of the trivial module, and the additive inverse of $[M]$ is $-[M] = [M^*]$. The group $T(G)$ is called the *group of endotrivial kG-modules*.

If $P_* \twoheadrightarrow M$ is a minimal projective resolution of a kG-module M, we denote by $\Omega^n(M)$ the $(n-1)$-st kernel of the differentials, and similarly $\Omega^{-n}(M)$ is the $(n-1)$-st cokernel in an injective resolution $M \hookrightarrow I_*$ of M. By convention, $\Omega^0(M)$ is the projective-free part of M and $\Omega(M) = \Omega^1(M)$. Thus part (6) in Lemma 2.2 provides us with an important family of endotrivial modules, namely the modules $\Omega^n(k)$ for $n \in \mathbb{Z}$. Henceforth we write Ω for the equivalence class of $\Omega(k)$ in $T(G)$. For $m, n \in \mathbb{Z}$, $\Omega^n(k) \otimes \Omega^m(k) \cong \Omega^{m+n}(k) \oplus (\text{proj})$, hence $\langle \Omega \rangle \leq T(G)$ is a cyclic subgroup, and, unless G has p-rank one, $\langle \Omega \rangle \cong \mathbb{Z}$.

2.2. **Previously known results.** The abelian group $T(G)$ is known to be finitely generated [CMN06, Cor. 2.5] and so we can write

$$T(G) = TT(G) \oplus TF(G)$$

where $TT(G)$ is the torsion subgroup of $T(G)$ and $TF(G)$ is a torsion-free direct sum complement of $TT(G)$ in $T(G)$. In particular $TT(G)$ is finite. The rank of $TF(G)$ is the *torsion-free rank* of $T(G)$ and it is written n_G. It is determined in [CMN06].

Lemma 2.4 ([CMN06])**.** *The torsion-free rank of $T(G)$ is equal to the number of conjugacy classes of maximal elementary abelian p-subgroups of rank 2 if G has p-rank 2, or that number plus one if G has p-rank greater than 2.*

The next three results characterising the torsion-free rank of $T(G)$ are at the crossroad of local group structure and representation theory.

Theorem 2.5 ([GM10, Mac70])**.** *Let G be a finite p-group. Suppose that G has a maximal elementary abelian p-subgroup of order p^2. Then G has p-rank at most p if p is odd, or at most 4 if $p = 2$.*

For completeness, the case $p = 2$ is due to A. MacWilliams (often coined the *four-generator theorem*), and for odd primes it is work of G. Glauberman and the second author.

Corollary 2.6. *Assume the p-rank of the group G is greater than p if p is odd, or greater than 4 if $p = 2$. Then $T(G)$ has torsion-free rank one.*

Theorem 2.7 ([Car07, Maz08])**.** *Let G be a finite p-group. Then $n_G \leq p + 1$ if p is odd, or at most $n_G \leq 5$ if $p = 2$. Moreover, both upper bounds are optimal.*

Optimality of the bound n_G for p odd is for instance obtained with G an extraspecial p-group of order p^3 and exponent p if p is odd, whereas for $p = 2$, we can take for G an extraspecial 2-group of order 32 of the form $Q_8 * D_8$.

We now present a collection of useful results, which have been proven in previous articles on endotrivial modules, see [Car12] and the references therein. Recall that a subgroup H of G is *strongly p-embedded* if p divides the order of H but p does not divide the order of $H \cap {}^g H$ for any $g \in G \setminus H$. In particular, $N_G(P)$ is strongly p-embedded in G if and only if the Sylow p-subgroup P is a trivial intersection (T.I. hereafter) subset of G. Also, if H is strongly p-embedded in G, then $H \geq N_G(P)$.

Useful information on $T(G)$ can sometimes be gathered by comparison with known endotrivial modules for subgroups of G. For instance, if H is a subgroup of G, the restriction along the inclusion induces a group homomorphism $\mathrm{Res}_H^G : T(G) \to T(H)$. In

contrast, the induction $\mathrm{Ind}_H^G(M)$ of an endotrivial kH-module M needs not be endotrivial for G.

Lemma 2.8 (Omnibus Lemma)**.** *Let H be a subgroup of G containing the normaliser $N_G(P)$ of a Sylow p-subgroup P of G.*

(1) *If G has p-rank at least 2, then Ω generates an infinite cyclic direct summand of $TF(G)$ and can be chosen as part of a set of generators for $TF(G)$. In particular, if $T(G)$ has torsion-free rank 1, then $TF(G) = \langle \Omega \rangle$.*

(2) *The restriction map $\mathrm{Res}_H^G : T(G) \longrightarrow T(H)$ is injective. More precisely, if M is an indecomposable endotrivial kG-module and if $M \downarrow_H^G = L \oplus (proj)$ where L is an indecomposable kH-module, then M is the kG-Green correspondent of L and L is endotrivial.*

(3) *Assume that for all $x \in G$ the subgroup ${}^x P \cap P$ is non trivial. Then, the kernel of the restriction map $\mathrm{Res}_P^G : T(G) \to T(P)$ is generated by the isomorphism classes of 1-dimensional kG-modules. In particular, this holds whenever G has a non-trivial normal p-subgroup.*

(4) *If H is strongly p-embedded in G, then $\mathrm{Res}_H^G : T(G) \longrightarrow T(H)$ is an isomorphism, the inverse map being induced by the induction Ind_H^G.*

(5) *If $P = N_G(P)$ is neither cyclic, nor semi-dihedral, nor generalised quaternion, then $TT(G) = \{[k]\}$.*

Lemma 2.8 shows that a large part of $TT(G)$ is generated by the (stable) isomorphism classes of trivial source endotrivial modules. Indeed, unless a Sylow p-subgroup P of G has normal p-rank one, then $T(P)$ is torsion-free and so $TT(G)$ is detected by restriction to P in the sense of Lemma 2.8(2).

Many of the Sylow p-subgroups of sporadic groups are cyclic for $p > 2$. In fact if G is quasi-simple with $G/Z(G)$ sporadic simple and $P \in \mathrm{Syl}_p(G)$ is cyclic, then $P \cong C_p$ and $p \neq 2$. So [MT07, Theorem 3.2] describes the structure of $T(G)$ in this case.

Theorem 2.9 ([MT07])**.** *Let G be a finite group with a non-trivial cyclic Sylow p-subgroup P with $|P| > 2$. Let Z be the unique subgroup of P of order p and let $H = N_G(Z)$. Then:*

(1) $T(G) = \{ [\mathrm{Ind}_H^G(M)] \mid [M] \in T(H) \} \cong T(H) \,.$

(2) *There is an exact sequence*

$$0 \longrightarrow X(H) \longrightarrow T(H) \xrightarrow{\mathrm{Res}_P^H} T(P) \longrightarrow 0$$

where $T(P) \cong \mathbb{Z}/2$. This sequence splits if and only if $[\Omega_H^2(k)]$ is a square in $X(H)$. In particular, this sequence splits if the index $e := |N_G(Z) : C_G(Z)|$ is odd.

2.3. **Character theory for endotrivial modules.** Finally we recall known results needed for our investigations which stem from ordinary character theory.

Theorem 2.10 ([LMS13, Theorem 1.3 and Corollary 2.3])**.** *Let $(K, \mathcal{O}, k)$ be a splitting p-modular system. Let V be an endotrivial kG-module. Then*

(1) *V is liftable to an endotrivial $\mathcal{O}G$-lattice.*

(2) *Moreover, if V is liftable to a $\mathbb{C}G$-module affording the character χ, then $|\chi(g)| = 1$ for every p-singular element $g \in G$.*

Further, if a kG-module M is the kG-Green correspondent of a 1-dimensional $kN_G(P)$-module, then M is a trivial source module and it lifts uniquely to a trivial source $\mathcal{O}G$-lattice $\hat{M}$. Denote by $\chi_{\hat{M}}$ the ordinary character afforded by $\hat{M}$. Then endotriviality for M can be read from the character table of G as follows:

Theorem 2.11 ([LM15, Theorem 2.2]). *Let $P \in Syl_p(G)$ and let M be the kG-Green correspondent of a 1-dimensional $kN_G(P)$-module. Then M is endotrivial if and only if $\chi_{\hat{M}}(x) = 1$ for all p-elements $x \in G \setminus \{1\}$.*

The latter proposition relies on the following result by Green, Landrock and Scott on character values of trivial source modules, which we will also implicitly extensively use in computations in Sections 4 and 6.

Lemma 2.12 ([Lan83, Part II, Lemma 12.6]). *Let M be an indecomposable trivial source kG-module and $x \in G$ a p-element. Then,*

  (1) $\chi_{\hat{M}}(x) \geq 0$ *is an integer (corresponding to the multiplicity of the trivial $k\langle x \rangle$-module as a direct summand of $M\downarrow^G_{\langle x \rangle}$);*

  (2) $\chi_{\hat{M}}(x) \neq 0$ *if and only if x belongs to a vertex of M.*

3. Sporadic groups and their covers

3.1. **Notation and terminology.** We call *sporadic simple* any of the 26 standard sporadic simple groups together with the Tits simple group $^2F_4(2)'$. We use the *Atlas* [CCN+85] notation, and in particular:

$A \times B$ denotes a direct product of groups A and B;

   $A.B$ or AB denotes a group having a normal subgroup isomorphic to A with corresponding quotient isomorphic to B;

 $A : B$ denotes a split extension of A by B;

 $A * B$ denotes a central product of A by B;

   m denotes a cyclic group of order m;

   p^n denotes an elementary abelian p-group of order p^n;

p_+^{1+2n} where p is an odd prime, denotes an extraspecial group of order p^{1+2n} and exponent p;

   D_{2^n} denotes a dihedral 2-group of order 2^n;

 SD_{2^n} denotes a semi-dihedral 2-group of order 2^n;

   Q_8 denotes the quaternion 2-group of order 8;

   $\mathfrak{S}_n$ denotes the symmetric group on n letters;

   $\mathfrak{A}_n$ denotes the alternating group on n letters.

If more than one symbol is used, we read them left to right, that is we write $A.B : C$ for $(A.B) : C$. In addition $Z(G)$ denotes the centre of G and $G' = [G, G]$ the commutator subgroup of G. Also, a p'-group for a prime p is a group of order not divisible by p.

3.2. **Covering groups and inflation.** A finite group G is called *quasi-simple* if G is a perfect central extension of a simple group. If G is quasi-simple, then $H := G/Z(G)$ is simple and G is a perfect central extension of H. We then call G a *covering group of H*. According to Notation 3.1, $m.H$ denotes a covering group of a group H, which is a central extension of H by a cyclic group of order m. The integer m is called the *degree* of

the central extension. If ℓ is a prime dividing m, then there exists a covering group K of H such that $m.H$ is a central extension of K of degree ℓ.

Lemma 3.1. *Let* $1 \longrightarrow A \longrightarrow G \overset{\pi}{\longrightarrow} H \longrightarrow 1$ *be a central extension of H by A, where G is a perfect group, and let P, Q be Sylow p-subgroups of G and H respectively. Then the following hold.*

(1) $X(N_H(Q))$ *is isomorphic to a quotient group of* $X(N_G(P))$.
(2) *If p divides $|A|$ and $TT(P) = \{[k]\}$, then $TT(G) = \{[k]\}$.*
(3) *If p does not divide $|A|$, then the torsion-free rank of $T(H)$ equals the torsion-free rank of $T(G)$.*

Recall that $T(P)$ is torsion-free if and only if P is neither cyclic, generalised quaternion nor semi-dihedral.

Proof.

(1) By assumption, π induces surjective homomorphisms $N_G(P) \overset{\pi}{\longrightarrow} N_H(Q)$ with kernel A, and $PN_G(P)' \overset{\pi}{\longrightarrow} QN_H(Q)'$ with kernel $O_p(A)(A \cap N_G(P)')$. So π induces a surjective homomorphism $N_G(P)/PN_G(P)' \twoheadrightarrow N_H(Q)/QN_H(Q)'$ with kernel isomorphic to $A/O_p(A)(A \cap N_G(P)')$. Therefore $X(N_H(Q))$ is isomorphic to a quotient of $X(N_G(P))$.
(2) Since $T(P)$ is torsion-free, then $TT(G)$ is generated by the classes of the trivial source endotrivial modules. Now since $p \,|\, |A|$, Lemma 2.8(3) implies that such module must have dimension 1, but because G is perfect, only the trivial module has dimension 1.
(3) The torsion-free ranks of $T(G)$ and $T(H)$ are determined by the number of conjugacy classes of maximal elementary abelian subgroups of rank 2. The claim follows from the fact that A is central in $G = A.H$.

$\square$

When investigating endotrivial modules we often rely on an operation called inflation. Namely, if N is a normal subgroup of the group G, the restriction along the natural projection map $G \to G/N$ makes a $k[G/N]$-module V into a kG-module $\mathrm{Inf}_{G/N}^G(V)$ on which N acts trivially. This operation is the *inflation* from G/N to G. If $p \nmid |N|$, then inflation of an endotrivial $k[G/N]$-module is an endotrivial kG-module. Moreover, if N is central, independently from the order of N, inflation commutes with Green correspondence in the following sense.

Lemma 3.2. *Let G be a finite group and let N be a normal subgroup of G.*

(1) *Let M be an endotrivial $k[G/N]$-module with $\dim_k(M) > 1$. Then $\mathrm{Inf}_{G/N}^G(M)$ is an endotrivial kG-module if and only if p does not divide $|N|$. In particular, if p does not divide $|N|$, then inflation induces a well-defined group homomorphism*

$$\mathrm{Inf}_{G/N}^G : T(G/N) \to T(G) : [M] \mapsto [\mathrm{Inf}_{G/N}^G(M)].$$

(2) *Assume that $N \leq Z(G)$ and let $P \in \mathrm{Syl}_p(G)$. Let V be a kG-module with vertex P such that $V = \mathrm{Inf}_{G/N}^G(\overline{V})$ for some $k[G/N]$-module $\overline{V}$. Let $\Gamma(-)$ and $\overline{\Gamma}(-)$*

denote the Green correspondence from G to $N_G(P)$ and from G/N to $N_G(P)/N$ respectively. Then $\mathrm{Inf}_{N_G(P)/N}^{N_G(P)}(\overline{\Gamma}(\overline{V})) \cong \Gamma(\mathrm{Inf}_{G/N}^G(\overline{V}))$.

Proof.

   (1) Set $H := G/N$ and let P, Q be Sylow p-subgroups of G and H respectively. If p divides $|N|$, then non-trivial p-subgroups of N acts trivially on $\mathrm{Inf}_H^G(L)$ for every projective kH-module L. Therefore it follows by definition that the inflation $\mathrm{Inf}_H^G(V)$ of an endotrivial kH-module V is not an endotrivial kG-module. Now suppose that p does not divide $|N|$, choose an isomorphism $\phi : P \to Q$ and denote by Res_ϕ the restriction along ϕ of a kQ-module. Then, because $\mathrm{Res}_P^G \mathrm{Inf}_H^G(M) \cong \mathrm{Res}_\phi \mathrm{Res}_Q^H(M)$ for any kH-module M, by Lemma 2.2(3) inflation preserves endotrivial modules. The claim follows.

   (2) See [DK09, Proposition 2.9]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.3. **T.I. Sylow p-subgroups.** We end this section with the list of the finite simple groups which have noncyclic trivial intersection (T.I.) Sylow p-subgroups. Recall that $P \subset G$ is T.I. if $P \cap {}^g P = \{1\}$ for any $g \in G \setminus N_G(P)$. In particular, if $|P|$ is prime, then P is T.I..

Theorem 3.3. [Mic91, Proposition 4.6] *Let G be a nonabelian simple group with a noncyclic T.I. Sylow p-subgroup P. Then G is isomorphic to one of the following groups.*

   (1) $\mathrm{L}_2(q)$ *where* $q = p^n$ *with* $n \geq 2$.
   (2) $\mathrm{U}_3(q)$ *where* q *is a power of* p.
   (3) ${}^2\mathrm{B}_2(2^{2m+1})$ *and* $p = 2$.
   (4) ${}^2\mathrm{G}_2(3^{2m+1})$ *with* $m \geq 1$ *and* $p = 3$.
   (5) $\mathrm{L}_3(4)$ *or* M_{11} *and* $p = 3$.
   (6) ${}^2\mathrm{F}_4(2)'$ *or* McL *and* $p = 5$.
   (7) J_4 *and* $p = 11$.

   We are now ready to combine both Sections 2 and 3 in order to investigate endotrivial modules for the sporadic groups and their covers. We start with the case $p = 2$ and whenever we think it more appropriate, we use the algebra softwares MAGMA [BCP97] or GAP [GAP13, Bre12]. For the clarity of exposition, we summarise the results in Table 3 and Table 5 in Section 7.

4. Endotrivial modules for sporadic groups in characteristic 2

   In this section, we prove the results stated in Section 7 for a field k of characteristic 2. The 2-local structure of simple groups has been thoroughly analysed. In particular, by a celebrated result due to Brauer and Suzuki it is well-known that a Sylow 2-subgroup of a simple group cannot be generalised quaternion ([BS59]), while [Suz86, Corollary 2, p. 144] asserts that a Sylow 2-subgroup of a nonabelian simple group cannot be cyclic. In view of these results it seems reasonable to believe that if G is a nonabelian simple group then $T(G)$ has torsion-free rank one in characteristic 2. We can prove slightly better.

Proposition 4.1. *Let G be a finite quasi-simple group with $G/Z(G)$ sporadic simple, then the torsion-free rank of $T(G)$ is one, that is, $TF(G) = \langle \Omega \rangle \cong \mathbb{Z}$.*

Proof. Using Theorems 2.5 and 2.7, we need to show that if G has a nonabelian Sylow 2-subgroup P of rank $2, 3$ or 4, then P is either semi-dihedral, or P has rank greater than 2 and no maximal elementary abelian subgroup of order 4. By inspection of [GLS98, Table 5.6.1], we are left with the following groups. If the 2-rank is 2, then $G = \mathrm{M}_{11}$ and has a semi-dihedral Sylow 2-subgroup of order 16. It follows from [CMT13, Thm. 6.5] that $n_G = 1$. Otherwise G is one of $\mathrm{M}_{12}, 2.\mathrm{M}_{12}, \mathrm{M}_{22}, 4.\mathrm{M}_{22}, 3.\mathrm{M}_{22}, 12.\mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{J}_1, \mathrm{J}_2, 2.\mathrm{J}_2,$ $\mathrm{J}_3, 3.\mathrm{J}_3, \mathrm{Co}_3, \mathrm{HS}, \mathrm{McL}, 3.\mathrm{McL}, \mathrm{Ly}, \mathrm{O'N}, 3.\mathrm{O'N}$ and its 2-rank is 3 or 4. A routine verification of each of these cases shows that a Sylow 2-subgroup of G has no maximal elementary abelian subgroup of order 4. $\qquad\square$

By Lemma 2.8 and the specific 2-local structure of sporadic simple groups and their covers, the torsion subgroup of $T(P)$ is often trivial, and the Sylow 2-subgroups are often self-normalising. This leads us to our next result.

Lemma 4.2. *If G is one of the groups* $\mathrm{M}_{12}, 2.\mathrm{M}_{12}, \mathrm{M}_{22}, 2.\mathrm{M}_{22}, 4.\mathrm{M}_{22}, 6.\mathrm{M}_{22}, 12.\mathrm{M}_{22},$ $2.\mathrm{J}_2, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{HS}, 2.\mathrm{HS}, \mathrm{McL}, \mathrm{He}, \mathrm{Ru}, 2.\mathrm{Ru}, \mathrm{O'N}, \mathrm{Co}_3, \mathrm{Co}_2, \mathrm{Fi}_{22}, 2.\mathrm{Fi}_{22}, 6.\mathrm{Fi}_{22}, \mathrm{Ly},$ $\mathrm{Th}, \mathrm{Fi}_{23}, \mathrm{Co}_1, 2.\mathrm{Co}_1, \mathrm{J}_4, \mathrm{Fi}'_{24}, \mathrm{B}, 2.\mathrm{B}, \mathrm{M},$ or $^2\mathrm{F}_4(2)',$ then $TT(G) = \{[k]\}.$*

Proof. Let $P \in \mathrm{Syl}_2(G)$. If G is one of $\mathrm{M}_{12}, 2.\mathrm{M}_{12}, \mathrm{M}_{22}, 2.\mathrm{M}_{22}, 4.\mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{HS},$ $2.\mathrm{HS}, \mathrm{McL}, \mathrm{He}, \mathrm{Ru}, 2.\mathrm{Ru}, 2.\mathrm{Suz}, 6.\mathrm{Suz}, \mathrm{O'N}, \mathrm{Co}_3, \mathrm{Co}_2, \mathrm{Fi}_{22}, 2.\mathrm{Fi}_{22}, 6.\mathrm{Fi}_{22}, \mathrm{Ly}, \mathrm{Th}, \mathrm{Fi}_{23},$ $\mathrm{Co}_1, 2.\mathrm{Co}_1, \mathrm{J}_4, \mathrm{Fi}'_{24}, \mathrm{B}, 2.\mathrm{B}, \mathrm{M},$ or $^2\mathrm{F}_4(2)',$ then one can read from [Wil98, Table 1] that P is self-normalising. Therefore, $TT(G) = \{[k]\}$ by Lemma 2.8. If G is one of $6.\mathrm{M}_{22},$ $12.\mathrm{M}_{22}, 2.\mathrm{J}_2, 2.\mathrm{Suz}, 6.\mathrm{Suz},$ or $6.\mathrm{Fi}_{22}$ then G has a non-trivial normal 2-subgroup and so $TT(G) = \{[k]\}$ by Lemma 3.1. $\qquad\square$

Lemma 4.3. *If G is one of the 3-fold covers* $3.\mathrm{M}_{22}, 3.\mathrm{Suz}, 3.\mathrm{O'N}, 3.\mathrm{Fi}_{22},$ or $3.\mathrm{Fi}'_{24},$ then G has no faithful endotrivial module. Therefore $T(G) \cong T(G/Z(G)) \cong \mathbb{Z}$ via inflation.*

Proof. Assume M is a faithful endotrivial kG-module and let χ be the ordinary character of G belonging to a lift of M. Then all irreducible constituents of χ lie in a common faithful block of kG and by Theorem 2.10 the values of χ on 2-singular conjugacy classes are roots of unity. However, it can be read from the GAP Character Table Libraries [Bre12] (or the *Atlas* [CCN+85]) that G has a 2-singular class C such that for all faithful characters $\psi \in \mathrm{Irr}(G)$ we have $\psi(c) \in m\mathbb{Z}$ with $m \in \mathbb{Z} \setminus \{\pm 1\},$ $c \in C,$ hence a contradiction. More precisely for $G = 3.\mathrm{M}_{22},$ take $(m, C) = (2, 6c),$ for $G = 3.\mathrm{Suz}$ take $(m, C) = (0, 6q),$ for $G = 3.\mathrm{O'N}$ take $(m, C) = (0, 6c),$ for $G = 3.\mathrm{Fi}_{22}$ take $(m, C) = (0, 6w),$ for $G = 3.\mathrm{Fi}'_{24}$ take $(m, C) = (0, 6ad).$ (Notation for conjugacy classes is that of GAP [Bre12].) $\qquad\square$

We are left with the computation of the torsion subgroup $TT(G)$ in characteristic 2 for the following groups: $\mathrm{M}_{11}, \mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3, 3.\mathrm{J}_3, 3.\mathrm{McL}, \mathrm{Suz}, \mathrm{HN}.$ We proceed case by case.

4.1. The Mathieu group M_{11}.

The group $G = \mathrm{M}_{11}$ has a semi-dihedral Sylow 2-subgroup P of order 16 and $P = N_G(P).$ From [CMT13, Theorems 6.4 and 6.5], we know that $T(\mathrm{M}_{11}) = \langle \Omega, [M] \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2$ where M is the kG-Green correspondent of the translated relative syzygy $\Omega(\Omega_{P/C}(k))$ for a noncentral subgroup C of P of order 2. Moreover M is self-dual. This module was explicitly described by Kawata and Okuyama [Kaw93, Thm. 5.1] as the relative syzygy $M \cong \Omega(\Omega_{G/C}(k)).$ Computations with MAGMA show that $\dim_k(M) = 41.$

4.2. **The Janko groups:** J_1**,** J_2**,** J_3 **and** $3.J_3$**.**

Let $G = J_1$ and P a Sylow 2-subgroup of G. Then $N_G(P) \cong P : 7 : 3$ (cf [CCN$^+$85]) and $X(N_G(P)) \cong \mathbb{Z}/3$ is generated by a 1-dimensional kN-module λ with $\lambda^{\otimes 3} = k$. The socle and Loewy series of the kG-Green correspondent $\Gamma(\lambda)$ of λ have been computed in [LM78, Cor 5.8], from which we gather that $\Gamma(\lambda)$ has dimension 133 and affords the ordinary character χ_{12}. By Lemma 2.2(7), $\Gamma(\lambda)$ cannot be endotrivial as $133 \not\equiv 1 \pmod{|P|}$. Therefore $TT(J_1) = \{[k]\}$.

The groups J_2 and J_3 have very similar 2-local structure. Put G for any of these Janko groups. The group G has a Sylow 2-subgroup $P = (Q_8 * D_8) : 2^2$ of order 2^7 and 2-rank 4. The index of P in its normaliser N is 3, so that $X(N) \cong \mathbb{Z}/3$. Moreover G has a maximal subgroup H containing N and H is isomorphic to $(Q_8 * D_8) : \mathfrak{A}_5$. The permutation module $k[H/P]$ gives all the indecomposable trivial source kH-modules up to isomorphism and MAGMA computations give that $k[H/P]$ splits into a direct sum of modules with dimensions $1, 4, 5, 5$. Therefore k is the only torsion endotrivial kH-module and a fortiori $TT(G) = \{[k]\}$ for $G = J_2$ and $G = J_3$. For $G = 3.J_3$ we read from [CCN$^+$85] that G has a maximal subgroup $C(2A)$ which is the centraliser of the central involution of P. Using MAGMA, we obtain that $C(2A) \cong 3 \times (Q_8 * D_8).\mathfrak{A}_5$ and its derived subgroup is $(Q_8 * D_8).\mathfrak{A}_5$. Therefore $X(C(2A)) \cong \mathbb{Z}/3$ and $TT(C(2A)) \cong \mathbb{Z}/3$. Referring the reader to the method developed in [Bal13], as explained in Section 6, a further MAGMA computation shows that $TT(3.J_3) \cong \mathbb{Z}/3$. In other words the Green correspondents of the two non-trivial 1-dimensional $k[C(2A)]$-modules are endotrivial.

4.3. **The group** $3.McL$**.** Let $G = 3.McL$ and P a Sylow 2-subgroup of G. Then $|P| = 2^7$ and we have $X(N_G(P)) \cong \mathbb{Z}/3$ since a Sylow 2-subgroup of McL is self-normalising. Because $TT(McL) = \{[k]\}$ by Lemma 4.2, a non-trivial trivial source endotrivial kG-module must be faithful. We see from the GAP Character Table Libraries [Bre12] that G has three 2-blocks of full defect: the principal block B_0 and two faithful blocks B_7 and B_8. However all simple modules in B_7 and B_8 have even dimension, so that by Lemma 2.2(7) none of these blocks can contain endotrivial modules, whence $TT(3.McL) = \{[k]\}$.

4.4. **The group** Suz**.**

The group Suz has a Sylow 2-subgroup P of order 2^{13}, rank 6 and index 3 in its normaliser N, so that $X(N) \cong \mathbb{Z}/3$. Furthermore N is contained in a maximal subgroup H of Suz of the form $H = 2^{1+6}_-.U_4(2)$ and $|H : N| = 135$. Let $1_{10} \in X(N)$ be the 10-th linear character of N (according to the GAP character tables [Bre12] notation), which has order 3. Then

$$\mathrm{Ind}_N^H(1_{10}) = \chi_8 + \chi_{12} + \chi_{13} + \chi_{18},$$

where $\chi_8, \chi_{12}, \chi_{13}, \chi_{18} \in \mathrm{Irr}(H)$ have degree $15, 30, 30$ and 60 respectively. Since all four of these characters have classes $2a$ and $2b$ in their kernels, the kH-Green correspondent $\Gamma_H(1_{10})$ cannot be endotrivial by Theorem 2.11. Therefore, neither is the kG-Green correspondent $\Gamma_G(1_{10})$ and we obtain $TT(Suz) = \{[k]\}$.

4.5. **The group** HN **(also denoted** F_5 **or** F_{5+} **).**

The group $G = HN$ has a Sylow 2-subgroup P of order 2^{14} and G has two maximal subgroups which are centralisers of involutions. We gather from [CCN$^+$85] that the centraliser of a 2-central involution has the form $H = 2^{1+8}_+.(\mathfrak{A}_5 \times \mathfrak{A}_5).2$. So H must contain

$N_G(P)$ and we have $TT(H) = X(H)$ since H has a non-trivial normal 2-subgroup. Now $X(H) = H/PH' = 1$ because $H' \geq (\mathfrak{A}_5 \times \mathfrak{A}_5)' = \mathfrak{A}_5 \times \mathfrak{A}_5$. Therefore, by injectivity of the restriction map $T(G) \to T(H)$ we conclude that $TT(G) = \{[k]\}$.

5. The torsion-free rank of $TF(G)$ in odd characteristic and p-rank ≥ 2

Throughout this section, we assume that p is odd. We compute the torsion-free rank n_G of $T(G)$, for G a quasi-simple group such that $G/Z(G)$ is sporadic simple and has a noncyclic Sylow p-subgroup. Recall from Lemma 2.4 that n_G is equal to the number of conjugacy classes of maximal elementary abelian p-subgroups of rank 2 if G has p-rank 2, or that number plus one if G has p-rank greater than 2.

5.1. **Sporadic groups with p-rank** 2. First we assume that the p-rank is exactly 2. If $P \in \mathrm{Syl}_p(G)$, then a Sylow p-subgroup P of G is either elementary abelian of rank 2, or P is an extraspecial p-group of order p^3 and exponent p (see [GLS98]).

Suppose that P is elementary abelian. Then $n_G = 1$ and $TF(G) = \langle \Omega \rangle \cong \mathbb{Z}$. This occurs in the following cases:

characteristic 3, $\quad M_{11}$, M_{22} , $2.M_{22}$, $4.M_{22}$, M_{23} , HS , $2.$HS,

characteristic 5, $\quad J_2$, $2.J_2$, He , Suz , $2.$Suz , $3.$Suz , $6.$Suz ,

$\qquad\qquad\qquad Fi_{22}$, $2.Fi_{22}$, $3.Fi_{22}$, $6.Fi_{22}$, Fi_{23} , Fi'_{24} , $3.Fi'_{24}$, $^2F_4(2)'$,

characteristic 7, $\quad Co_1$, $2.Co_1$, Th , B , $2.$B,

characteristic 11, $\quad$M .

Suppose now that P is extraspecial of order p^3 and exponent p, that is $P = p_+^{1+2}$ according to Notation 3.1. We use the presentation

$$P = \langle x, y, z \mid x^p = y^p = z^p = [x,z] = [y,z] = 1, [x,y] = z \rangle$$

where $[x,y] = xyx^{-1}y^{-1}$. Then $Z(P) = P' = \Phi(P) = \langle z \rangle$ has order p, where $\Phi(P)$ is the Frattini subgroup. The outer automorphism group of P is isomorphic to $\mathrm{GL}_2(p)$ and the matrix $\left(\begin{smallmatrix} r' & r \\ s' & s \end{smallmatrix} \right)$ with determinant d corresponds to the class of automorphisms sending x to $x^{r'}y^{s'}$, y to $x^r y^s$ and z to z^d.

Ruiz and Viruel [RV04] have classified the saturated fusion systems on P providing us with the information we need in order to compute n_G. That is, the number of G-conjugacy classes of the $p+1$ elementary abelian p-subgroups of rank 2. We refer the reader to [RV04] for details, and only give the results that apply in our context specifically. We recall that a p-subgroup Q is p-centric in G if $Z(Q)$ is a Sylow p-subgroup of $C_G(Q)$, and Q is p-radical if $N_G(Q)/QC_G(Q)$ has no non-trivial normal p-subgroup. Moreover the set of p-radical p-centric subgroups of $P = p_+^{1+2}$ is closed under G-conjugation for any finite group G, and if this set is empty, then the p-fusion is controlled by the normaliser of P.

Proposition 5.1. *Let p be an odd prime and G be a quasi-simple group such that $G/Z(G)$ is sporadic simple. If the p-rank of G is 2, then the torsion-free part of $T(G)$ is as given in Table 1.*

TABLE 1. $TF(G)$ for sporadic quasi-simple groups in p-rank 2, $p > 2$

G	P	$TF(G)$	G	P	$TF(G)$	G	P	$TF(G)$
M_{11}	3^2	$\mathbb{Z}$	McL	5_+^{1+2}	$\mathbb{Z}$	$3.Fi_{22}$	5^2	$\mathbb{Z}$
M_{12}	3_+^{1+2}	$\mathbb{Z}^3$	$3.McL$	5_+^{1+2}	$\mathbb{Z}$	$6.Fi_{22}$	5^2	$\mathbb{Z}$
$2.M_{12}$	3_+^{1+2}	$\mathbb{Z}^3$	He	3_+^{1+2}	$\mathbb{Z}^2$	Th	5_+^{1+2}	$\mathbb{Z}$
M_{22}	3^2	$\mathbb{Z}$	He	5^2	$\mathbb{Z}$	Th	7^2	$\mathbb{Z}$
$2.M_{22}$	3^2	$\mathbb{Z}$	He	7_+^{1+2}	$\mathbb{Z}^3$	Fi_{23}	5^2	$\mathbb{Z}$
$3.M_{22}$	3_+^{1+2}	$\mathbb{Z}$	Ru	3_+^{1+2}	$\mathbb{Z}$	Co_1	7^2	$\mathbb{Z}$
$4.M_{22}$	3^2	$\mathbb{Z}$	$2.Ru$	3_+^{1+2}	$\mathbb{Z}$	$2.Co_1$	7^2	$\mathbb{Z}$
$6.M_{22}$	3_+^{1+2}	$\mathbb{Z}$	Ru	5_+^{1+2}	$\mathbb{Z}^2$	J_4	3_+^{1+2}	$\mathbb{Z}$
$12.M_{22}$	3_+^{1+2}	$\mathbb{Z}$	$2.Ru$	5_+^{1+2}	$\mathbb{Z}^2$	J_4	11_+^{1+2}	$\mathbb{Z}$
J_2	3_+^{1+2}	$\mathbb{Z}$	Suz	5^2	$\mathbb{Z}$	Fi_{24}'	5^2	$\mathbb{Z}$
$2.J_2$	3_+^{1+2}	$\mathbb{Z}$	$2.Suz$	5^2	$\mathbb{Z}$	$3.Fi_{24}'$	5^2	$\mathbb{Z}$
J_2	5^2	$\mathbb{Z}$	$3.Suz$	5^2	$\mathbb{Z}$	Fi_{24}'	7_+^{1+2}	$\mathbb{Z}^3$
$2.J_2$	5^2	$\mathbb{Z}$	$6.Suz$	5^2	$\mathbb{Z}$	$3.Fi_{24}'$	7_+^{1+2}	$\mathbb{Z}^3$
M_{23}	3^2	$\mathbb{Z}$	O'N	7_+^{1+2}	$\mathbb{Z}^3$	B	7^2	$\mathbb{Z}$
HS	3^2	$\mathbb{Z}$	$3.O'N$	7_+^{1+2}	$\mathbb{Z}^3$	$2.B$	7^2	$\mathbb{Z}$
$2.HS$	3^2	$\mathbb{Z}$	Co_3	5_+^{1+2}	$\mathbb{Z}$	M	11^2	$\mathbb{Z}$
HS	5_+^{1+2}	$\mathbb{Z}^2$	Co_2	5_+^{1+2}	$\mathbb{Z}$	M	13_+^{1+2}	$\mathbb{Z}^2$
$2.HS$	5_+^{1+2}	$\mathbb{Z}^2$	Fi_{22}	5^2	$\mathbb{Z}$	$^2F_4(2)'$	3_+^{1+2}	$\mathbb{Z}^2$
M_{24}	3_+^{1+2}	$\mathbb{Z}^2$	$2.Fi_{22}$	5^2	$\mathbb{Z}$	$^2F_4(2)'$	5^2	$\mathbb{Z}$

Proof. By Lemma 2.4, the rank of $TF(G)$ is equal to n_G. Let P be a Sylow p-subgroup of G with p-rank 2.

If P is elementary abelian, then $n_G = 1$ and $TF(G) = \langle \Omega \rangle \cong \mathbb{Z}$ in the 28 cases listed above.

Now assume $P \cong p_+^{1+2}$. So P has $p+1$ elementary abelian p-subgroups of rank 2 and n_G is the number of G-conjugacy classes of such subgroups. We run through the possibilities for p and G according to the classification. The results mentioned below are taken or deduced from [RV04, Table 1.1, Table 1.2, Remark 1.4]. For convenience, let us write $\mathcal{E}$ for the set of the elementary abelian p-subgroups of rank 2 of P.

The case $p = 3$. In this case $|\mathcal{E}| = 4$.

If $G \in \{Ru, J_4\}$, then all four subgroups in $\mathcal{E}$ are G-conjugate, giving $n_G = 1$.

If $G = {}^2F_4(2)'$, then the subgroups in $\mathcal{E}$ split into two conjugacy classes, giving $n_G = 2$.

If $G \in \{M_{12}, 2.M_{12}\}$, then fusion in G is the same as in $L_3(3)$ so that $\mathcal{E}$ splits into three conjugacy classes, giving $n_G = 3$. More precisely, $\{\langle x, z \rangle\}$ and $\{\langle y, z \rangle\}$ are two G-conjugacy classes and both subgroups are 3-centric 3-radical. Moreover $N_G(P)/PC_G(P) \cong 2^2$ can be realised as the subgroup of diagonal matrices of $GL_2(3)$, so that $\langle xy, z \rangle$ and $\langle xy^2, z \rangle$ become conjugate via the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix} \right)$.

If $G \in \{3.M_{22}, 6.M_{22}, 12.M_{22}\}$, then 3 divides the order of the centre of G, and because all the subgroups of order 3 of a Sylow 3-subgroup of M_{22} are conjugate in M_{22} (cf

[CCN$^+$85]), the four subgroups of $\mathcal{E}$ in each of the above covers of M$_{22}$ are G-conjugate. Therefore $n_G = 1$.

If $G \in \{$M$_{24}$, He$\}$, then $\mathcal{E}$ splits into two conjugacy classes of size two (cf [RV04, Proof of Lemma 4.8]), one of which is formed by the two 3-centric 3-radical subgroups of P in G. So $n_G = 2$.

If $G = $ J$_2$, then all four subgroups in $\mathcal{E}$ are G-conjugate, giving $n_G = 1$. Indeed, there is no 3-centric 3-radical subgroup and $N_G(P)/PC_G(P) \cong 8$ can be realised as the subgroup of GL$_2(3)$ generated by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix} \right)$, which sends x to xy^2, y to xy and z to z^2. In consequence all four elements of $\mathcal{E}$ are conjugate.

The case $p = 5$. In this case $|\mathcal{E}| = 6$.

If $G = $ Th, then all six subgroups in $\mathcal{E}$ are G-conjugate, giving $n_G = 1$.

If $G \in \{$Ru, 2.Ru$\}$, then $\mathcal{E}$ splits into two conjugacy classes of size two and four (cf [RV04, Proof of Lemma 4.8]). The conjugacy class of size 2 is formed by the 5-centric 5-radical subgroups of P in G, and the other four are conjugate in $N_G(P)$. Note indeed that $N_G(P)/PC_G(P) \cong 4^2 : 2$.

In all the remaining cases [RV04] show that there are no 5-centric 5-radical subgroups of P in G, saying that the p-fusion is controlled by $N_G(P)$. In other words the G- and $N_G(P)$-conjugacy classes of the subgroups in $\mathcal{E}$ coincide, and their number gives us n_G. We split our analysis into four, according to the structure of $N_G(P)/PC_G(P)$ as a subgroup of GL$_2(5)$.

If $G = $ Co$_3$, then $N_G(P)/PC_G(P) \cong D_{48}$ and all six subgroups in $\mathcal{E}$ are G-conjugate. Therefore $n_G = 1$.

If $G = $ Co$_2$, then $N_G(P)/PC_G(P) \cong 4.\mathfrak{S}_4$ and all six subgroups in $\mathcal{E}$ are G-conjugate. Therefore $n_G = 1$.

If $G \in \{$HS, 2.HS$\}$, then $N_G(P)/PC_G(P) \cong D_{16}$ and we calculate that $\mathcal{E}$ splits into two G-conjugacy classes. Therefore $n_G = 2$.

If $G \in \{$McL, 3.McL$\}$, then $N_G(P)/PC_G(P) \cong 3 : 8$ and all six subgroups in $\mathcal{E}$ are G-conjugate. Therefore $n_G = 1$.

The case $p = 7$. In this case $|\mathcal{E}| = 8$. We split the analysis into three according to the structure of $N_G(P)/PC_G(P)$, which is explicitly described as a subgroup of GL$_2(7)$ in [RV04, Proof of Lemma 4.16] allowing us to compute n_G by hand. There are five groups to consider.

If $G = $ He, then $N_G(P)/PC_G(P) \cong \mathfrak{S}_3 \times 3$ and $\mathcal{E}$ splits into three conjugacy classes: two with three subgroups and one of size two, giving $n_G = 3$. In this case there are three G-conjugate 7-centric 7-radical subgroups.

If $G \in \{$Fi$'_{24}$, 3.Fi$'_{24}\}$, then $N_G(P)/PC_G(P) \cong \mathfrak{S}_3 \times 6$ and $\mathcal{E}$ splits into three conjugacy classes: two with three subgroups and one of size two, giving $n_G = 3$. In this case there are six 7-centric 7-radical subgroups split into two G-conjugacy classes of size 3.

If $G \in \{$O'N, 3.O'N$\}$, then $N_G(P)/PC_G(P) \cong D_8 \times 3$ and $\mathcal{E}$ splits into three conjugacy classes: two with two subgroups and one of size four, giving $n_G = 3$. In this case there are four 7-centric 7-radical subgroups split into two G-conjugacy classes of size 2.

The case $p = 11$. In this case $G = $ J$_4$ and Theorem 3.3 says that P is T.I., which implies $N_G(P) = N_G(Z(P))$. In this case there are no 11-centric 11-radical subgroups, and so the 11-fusion is controlled by $N_G(P)$. From [Ben81, CCN$^+$85], we get that $N_G(P) = $

$11_+^{1+2} : (5 \times (2.\mathfrak{S}_4))$ and $N_G(P)/PC_G(P) \cong 5 \times (2.\mathfrak{S}_4)$. Hence the twelve elementary abelian 11-subgroups of P of rank two are G-conjugate, and so $n_G = 1$.

The case $p = 13$. In this case $G = $ M and the fourteen elementary abelian 13-subgroups of P of rank two are split into two G-conjugacy classes, namely one of size six, consisting of 13-centric 13-radical subgroups, and another of size eight, giving $n_G = 2$. Here $N_G(P)/PC_G(P) \cong 3 \times (4.\mathfrak{S}_4)$ (cf [RV04, Proof of Lemma 4.18]). $\square$

5.2. **Groups with p-rank greater or equal to** 3.

Proposition 5.2. *Let G be a quasi-simple group such that $G/Z(G)$ is sporadic simple and let $p = char(k) = 3$. If G is one of the groups* $3.\mathrm{J}_3$, McL, $3.$McL, Suz, $2.$Suz, $3.$Suz, O'N, Co_3, Co_2, Fi_{22}, $2.\mathrm{Fi}_{22}$, $3.\mathrm{Fi}_{22}$, HN, Ly, Th, Fi_{23}, Co_1, $2.\mathrm{Co}_1$, Fi_{24}', $3.\mathrm{Fi}_{24}'$, B, $2.$B, M, *then* $TF(G) = \langle \Omega \rangle \cong \mathbb{Z}$.

Proof. In all cases the 3-rank of G is greater or equal to 4, see [GLS98, Table 5.6.1]. Therefore $n_G = 1$ by Theorem 2.5 and the claim follows from Lemma 2.4. $\square$

The remaining cases are the quasi-simple groups G such that $G/Z(G)$ is sporadic simple and G has p-rank between 3 and p.

For convenience, we introduce a complementary notation taken from [CCN+85]. Namely, we write nX^e, where n, e are positive integers and X a capital letter, to denote an elementary abelian group n^e spanned by elements belonging to the G-conjugacy class labelled nX in [CCN+85].

Proposition 5.3. *Let $p \geq 3$ and G be a quasi-simple group such that $G/Z(G)$ is sporadic simple. If the p-rank of G is greater or equal to 3 and smaller or equal to p, then G and $TF(G)$ are as listed in Table 2.*

TABLE 2. Sporadic quasi-simple groups with p-rank between 3 and p.

G	p	p-rank	$TF(G)$	G	p	p-rank	$TF(G)$	G	p	p-rank	$TF(G)$
J_3	3	3	$\mathbb{Z}$	Ly	5	3	$\mathbb{Z}$	M	5	4	$\mathbb{Z}$
$3.\mathrm{O'N}$	3	3	$\mathbb{Z}$	HN	5	3	$\mathbb{Z}$	M	7	3	$\mathbb{Z}$
Co_1	5	3	$\mathbb{Z}^2$	B	5	3	$\mathbb{Z}$				
$2.\mathrm{Co}_1$	5	3	$\mathbb{Z}^2$	$2.$B	5	3	$\mathbb{Z}$				

Proof. A Sylow 3-subgroup of J_3 has a non-cyclic center and therefore J_3 cannot have a maximal elementary abelian 3-subgroup of rank 2. For $3.\mathrm{O'N}$, Co_1 and $2.\mathrm{Co}_1$, we used MAGMA [BCP97] to calculate the number of conjugacy classes of the elementary abelian p-subgroups of rank 2, for the appropriate prime p, and then invoked Lemma 2.4. For the remaining cases, we gather information from [CCN+85, GLS98]. Let P be a Sylow p-subgroup of G. If $G = $ Ly and $p = 5$ then P is contained in the normaliser $N_G(5A^3) = 5^3.\mathrm{L}_3(5)$ of an elementary abelian subgroup $5A^3$. So $\mathrm{L}_3(5)$ acts as automorphism group of a 3-dimensional vector space $\mathbb{F}_5^3$ and by linear algebra, using Jordan forms, any 5-element of $\mathrm{L}_3(5)$ must centralise a non-trivial subgroup in 5^3 (i.e. must have a non-trivial eigenspace on $\mathbb{F}_5^3$). Therefore each element of order 5 in P has a centraliser in P of rank at

least 3, saying that P has no maximal elementary abelian subgroup of rank 2. A similar argument applies to B and 2.B, as B has a local subgroup of the form $N_G(5B^3) = 5^3 . \, \mathrm{L}_3(5)$. If $G = \mathrm{HN}$ and $p = 5$ we look at the normalisers of the elements of order 5 as given in [GLS98]. In the Sylow 5-subgroup of $N_G(5A) = (D_{10} \times \mathrm{U}_3(5)) : 2$ each element of order 5 is centralised by a subgroup of 5-rank at least 3. From $N_G(5B)$ we see that $C_P(5B)$ has rank at least 3 and similarly for $\overline{5C}$ and $5E$. Consequently G cannot have a maximal elementary abelian 5-subgroup of rank 2. Finally, if $G = \mathrm{M}$ and $p = 5$ then P is a Sylow 5-subgroup of a local subgroup of the form $5^{1+6}_+.(2.\,\mathrm{J}_2).4$ which has no maximal elementary abelian subgroups of rank 2 and so neither does M. If $p = 7$ then there are two conjugacy classes of elements of order 7. The normalisers have the form $((7.3) \times \mathrm{He}).2$ and $7^{1+4}_+.(2.\mathfrak{A}_7 \times 3).2$ and none of these has a maximal elementary abelian 7-subgroup of rank 2. $\qquad\square$

6. The torsion subgroup $TT(G)$ of $T(G)$ in odd characteristic.

Let p be an odd prime. We consider in this section groups G with a noncyclic Sylow p-subgroup P. (If P is cyclic, then we refer to Section 7.) By Lemma 2.8, the torsion subgroup $TT(G)$ identifies with a subgroup of $X(N_G(P))$ via restriction and consists precisely of the kG-Green correspondents of the 1-dimensional $kN_G(P)$-modules that are endotrivial, i.e. the trivial source endotrivial modules. Recall that for a group G we write $X(G)$ for the group of isomorphism classes of 1-dimensional kG-modules, and it is isomorphic to the p'-group $G/G'P$ where $G' = [G,G]$ is the commutator subgroup of G.

Simple endotrivial modules for covering groups of sporadic groups have been determined in [LMS13, §7], where the authors prove that many of them are trivial source modules, hence elements of $T(G)$. We now continue our analysis of $TT(G)$ along similar lines, using Green correspondence and induction and restriction of characters.

In the rest of this section we use the following notation. If G is a given sporadic group and $\ell.G$ an ℓ-fold cover of G, we write P and $\widetilde{P}$ for a Sylow p-subgroup of G and of $\ell.G$ respectively, and set $N = N_G(P)$ and $\widetilde{N} = N_{\ell.G}(\widetilde{P})$. Also, for a subgroup H of G containing N and for an indecomposable kN-module V, then $\Gamma_H(V)$ denotes the kH-Green correspondent of V. Similarly for $\ell.G$.

We use the structure of the Sylow normalisers and maximal subgroups as given in [CCN+85, GL83, GLS98] and the GAP Character Table Libraries [Bre12]. These are especially useful in determining $X(N)$. In our computations we denote by χ ordinary irreducible characters of a group. We index the ordinary irreducible characters and the blocks according to the GAP Character Table Libraries [Bre12], with the exception that we denote the principal block by B_0 instead of B_1. Accordingly we denote by e_i the central primitive idempotent corresponding to the block B_i. We denote by $1_H \in \mathrm{Irr}(H)$ the trivial character of a group H and $1_i \in X(H)$ denotes a non-trivial 1-dimensional kH-module. We also identify these with the corresponding ordinary and Brauer characters, as well as the corresponding simple kG-module and its class in $X(H)$.

Throughout we use the dimensional criterion for trivial source endotrivial modules of Lemma 2.2(7), the possible character values for characters of endotrivial modules given by Theorem 2.10 and Theorem 2.11, and the possible values of characters of trivial source modules given by Lemma 2.12. We split the cases according to the values of p and the

arguments we used to obtain $TT(G)$.

First, as for the 2-fold covers in characteristic 2, the 3-fold covers in characteristic 3 cannot have non-trivial torsion endotrivial modules.

Lemma 6.1. *Let $p = 3$ and let G be one of the covering groups* $3.\mathrm{M}_{22}$, $6.\mathrm{M}_{22}$, $12.\mathrm{M}_{22}$, $3.\mathrm{J}_3$, $3.\mathrm{McL}$, $3.\mathrm{Suz}$, $6.\mathrm{Suz}$, $3.\mathrm{O'N}$, $3.\mathrm{Fi}_{22}$, $6.\mathrm{Fi}_{22}$, $3.\mathrm{Fi}'_{24}$. *Then $TT(G) = \{[k]\}$.*

Proof. In all cases G has a non-trivial normal 3-subgroup so that $TT(G) = \{[k]\}$ by Lemma 3.1(2). ☐

Next we treat covering groups G such that $p \nmid |Z(G)|$.

Lemma 6.2. *Assume $p = 3$ and G is one of the covering groups* $2.\mathrm{M}_{12}$, $4.\mathrm{M}_{22}$, $2.\mathrm{J}_2$, $2.\mathrm{HS}$, $2.\mathrm{Suz}$, $2.\mathrm{Fi}_{22}$, $2.\mathrm{Co}_1$, $2.\mathrm{B}$, *or $p = 5$ and G is one of the covering groups* $2.\mathrm{J}_2$, $2.\mathrm{HS}$, $2.\mathrm{Ru}$, $2.\mathrm{Suz}$, $3.\mathrm{Suz}$, $6.\mathrm{Suz}$, $2.\mathrm{Co}_1$, $3.\mathrm{Fi}'_{24}$, $2.\mathrm{B}$, *or $p = 7$ and G is one of the covering groups* $2.\mathrm{Co}_1$, $3.\mathrm{Fi}'_{24}$, $2.\mathrm{B}$. *Then G has no faithful endotrivial module. Furthermore in all cases $T(G) \cong T(G/Z(G))$ via inflation, except if $G = 4.\mathrm{M}_{22}$ in which case $T(G) \cong T(2.\mathrm{M}_{22})$.*

Proof. Since $p \nmid |Z(G)|$ faithful endotrivial modules must afford characters in faithful blocks. The proof is similar to that of Lemma 4.3. Except for $2.\mathrm{HS}$ in characteristic 3, [Bre12] shows in all cases that G has a p-singular conjugacy class C such that for all faithful characters $\psi \in \mathrm{Irr}(G)$ we have $\psi(c) \in m\mathbb{Z}$ with $m \in \mathbb{Z} \setminus \{\pm 1\}$, $c \in C$. Hence we obtain a contradiction with the possible values of characters of endotrivial modules given by Theorem 2.10. More accurately, we can choose m and C as follows. For $p = 3$ take $m = 0$ if G is one of $2.\mathrm{M}_{12}$, $4.\mathrm{M}_{22}$, $2.\mathrm{J}_2$, $2.\mathrm{Suz}$, $2.\mathrm{Fi}_{22}$, $2.\mathrm{Co}_1$, or $2.\mathrm{B}$, and for C choose $12a$, $6b$, $12a$, $12a$, $12a$, $12a$, $6c$ respectively. For $p = 3$ and $G = 2.\mathrm{M}_{12}$ take $(m, C) = (2, 6c)$. If $p = 5$ take $m = 0$ if G is one of $2.\mathrm{J}_2$, $2.\mathrm{HS}$, $2.\mathrm{Suz}$, $3.\mathrm{Suz}$, $6.\mathrm{Suz}$, $2.\mathrm{Co}_1$, or $2.\mathrm{B}$ and for C choose $20a$, $20a$, $20a$, $15e$, $15e$, $20a$, $20b$ respectively. For $p = 5$, if $G = 2.\mathrm{Ru}$ take $(m, C) = (5, 5b)$, and if $G = 3.\mathrm{Fi}'_{24}$ take $(m, C) = (5, 15c)$. If $p = 7$ take $(m, C) = (2, 7a)$ for $G = 2.\mathrm{Co}_1$, take $(m, C) = (3, 7a)$ for $G = 3.\mathrm{Fi}'_{24}$, and take $(m, C) = (0, 14b)$ for $G = 2.\mathrm{B}$. Finally if $G = 2.\mathrm{HS}$ and $p = 3$, then all faithful characters take value in $3i\mathbb{Z}$ on class $12a$, hence the result. (Notation for conjugacy classes is that of GAP.) The claim on $T(G)$ is straightforward from Lemma 3.2. ☐

We now go through the list of sporadic simple groups by increasing order, including non-trivial covers not treated by Lemmas 6.1 and 6.2 above, and compute the torsion subgroup $TT(G)$ for as many groups G and characteristics p as possible. In the cases when we do not obtain the full structure of $TT(G)$ we give bounds instead.

Let us point out that the results for the groups He and Suz in characteristic 5, hereafter mentioned as 'MAGMA computation', partly use a method recently developed by Paul Balmer [Bal13] and a subsequent algorithm by Carlson (private communication). Balmer's method determines the kernel of the restriction $TT(G) \rightarrow TT(H)$ to a subgroup H of G by showing that this kernel is isomorphic to the set of the so-called weak H-homomorphisms. In particular, for $H = P$ the set of weak P-homomorphisms is isomorphic to the subgroup of $TT(G)$ spanned by the trivial source endotrivial modules. We omit here the technicalities, to avoid further deviation from our present scope, referring the reader to [Bal13]. The main drawback of this method is that it cannot always be

used in practice, and when it does, then it often relies on tedious computer calculations. Both authors are thankful to Jon Carlson with his help in implementing an algorithm in MAGMA and running it on specific groups.

6.1. **The group $G = \mathrm{M}_{11}$ in characteristic** 3. By Theorem 3.3, the group G has an abelian T.I. Sylow 3-subgroup $P \cong 3^2$ and $N \cong 3^2 : SD_{16}$. Thus $X(N) \cong SD_{16}/(SD_{16})' \cong (\mathbb{Z}/2)^2$. It follows from Lemma 2.8(3),(4) and Proposition 5.1 that $TT(G) \cong X(N) \cong (\mathbb{Z}/2)^2$.

The indecomposable modules representing the classes in $TT(G)$ are the kG-Green correspondents of the modules in $X(N)$. Their Loewy and socle series, as well as the characters they afford, are described in [KW99a, (4.3)]. Their dimensions are 1, 55, 55 and 10.

6.2. **The group $G = \mathrm{M}_{12}$ and its double cover in characteristic** 3. In this case $P = 3_+^{1+2}$ and we have a chain of subgroups

$$P \le N \le N_2 \le \mathrm{M}_{12} \quad \text{where } N \cong 3_+^{1+2} : 2^2 \text{ and } N_2 \cong 3^2 : \mathrm{GL}_2(3).$$

So $X(N) \cong (\mathbb{Z}/2)^2$ and $X(N_2) \cong \mathbb{Z}/2$. Since $|N_2 : N| = 4$, induction from kN to kN_2 of a 1-dimensional kN-module is a module of dimension 4. Hence the kN_2-Green correspondents $\Gamma_{N_2}(1_N), \Gamma_{N_2}(1_1), \Gamma_{N_2}(1_2), \Gamma_{N_2}(1_3)$ of the four 1-dimensional modules $1_N, 1_1, 1_2, 1_3 \in X(N)$ have dimension $1, 1, 4$ and 4, respectively, because N_2 has exactly two non-isomorphic 1-dimensional modules. In particular we conclude that $\Gamma_{N_2}(1_2)$, $\Gamma_{N_2}(1_3)$ are not endotrivial, for $4 \not\equiv 1 \pmod{|P|}$. A fortiori $\Gamma_G(1_2), \Gamma_G(1_3)$ are not endotrivial either. It remains to prove that $\Gamma_G(1_1) \notin TT(G)$. Note that the principal block B_0 is the unique block of kG with full defect. The Loewy and socle series of $\Gamma_G(1_1)$ have been computed in [KW99b, 6.1]. In particular, if ψ_1 denotes the linear character corresponding to $\Gamma_{N_2}(1_1)$, then we see that $e_0 \cdot \mathrm{Ind}_{N_2}^G(\psi_1)$ yields that $\Gamma_G(1_1)$ is the trivial source module affording the character $\chi_9 + \chi_{13}$ of degree $175 \not\equiv 1 \pmod{|P|}$. Whence $\Gamma_G(1_1) \notin TT(G)$ and $TT(G) = \{[k]\}$.

6.3. **The group $G = \mathrm{M}_{22}$ and its covers in characteristic** 3. A Sylow 3-subgroup P of G is elementary abelian of order 9 and $N \cong 3^2 : Q_8$. So $X(N) \cong (\mathbb{Z}/2)^2$. In order to obtain $TT(G)$, we first consider the group $2.G = 2.\mathrm{M}_{22}$. A Sylow 3-subgroup $\widetilde{P}$ of $2.G$ is elementary abelian of order 9 and $X(\widetilde{N}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$. We gather from [LMS13, Thm. 7.1] and [LM15, Sec. 2] that $2.G$ has six simple trivial source endotrivial modules. Namely, the trivial module k, a self-dual module S_{55} of dimension 55 affording the character $\chi_5 \in \mathrm{Irr}(2.G)$, and four faithful modules: S_{10}, S_{10}', both of dimension 10, affording the characters $\chi_{13}, \chi_{14} \in \mathrm{Irr}(2.G)$, and $S_{154}, (S_{154})^*$ both of dimension 154, affording the characters $\chi_{19}, \chi_{20} \in \mathrm{Irr}(2.G)$. It follows directly that $TT(2.G) \cong X(\widetilde{N}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$. Furthermore, by Lemma 3.2(2), Green correspondence preserves the faithfulness of blocks, hence $TT(G) \cong (\mathbb{Z}/2)^2$. The group $T(4.\mathrm{M}_{22})$ is isomorphic to $T(2.\mathrm{M}_{22})$ via inflation by Lemma 6.2. However, notice that $TT(4.\mathrm{M}_{22}) \not\cong X(\widetilde{N}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8$ in this case. Finally the groups $TT(3.\mathrm{M}_{22}), TT(6.\mathrm{M}_{22})$ and $TT(12.\mathrm{M}_{22})$ are all trivial by Lemma 3.1(2).

6.4. **The group $G = \mathrm{J}_2$.**

Characteristic 3. In this case $P \cong 3_+^{1+2}$, $N \cong 3_+^{1+2} : 8$, $X(N) \cong \mathbb{Z}/8$ and $B_0(kG)$ is the unique 3-block with full defect. We pick 1_4 to be a character of order 4 of $X(N)$, then $e_0 \cdot \mathrm{Ind}_N^G(1_4) = \chi_{12} + \chi_{13} + 2\chi_{21}$. By the criteria on degrees and character values

$\Gamma_G(1_4)$ cannot be endotrivial. It follows that $TT(G)$ can at most be cyclic of order 2. Let 1_7 denote the linear character of order 2 of N. We have $e_0 \cdot \mathrm{Ind}_N^G(1_7) = \chi_4 + \chi_5 + \chi_{12} + 2\chi_{13} + \chi_{16} + \chi_{17} + \chi_{20} + 2\chi_{21}$. Again using the possible degrees and values of trivial source characters given by Lemma 2.12 it is easy to see that $\Gamma_G(1_7)$ must afford the character $\chi_4 + \chi_5 + \chi_{13}$, which takes value 1 on all the non-trivial 3-elements and is thus endotrivial Theorem 2.11. We conclude that $TT(G) \cong \mathbb{Z}/2$.

Characteristic 5. In this case P is elementary abelian of rank 2 and $N \cong 5^2 : D_{12}$ so that $X(N) \cong (\mathbb{Z}/2)^2$. The principal 5-block of kG is the unique block with full defect. Let $X(N) = \{1_N, 1_1, 1_3, 1_4\}$. Inducing the non-trivial characters in $X(N)$ we get the following. First $e_0 \cdot \mathrm{Ind}_N^G(1_1) = \chi_{14} + \chi_{15} + \chi_{19}$, so that by the degree and character value criteria we obtain that $\Gamma_G(1_1)$ affords $e_0 \cdot \mathrm{Ind}_N^G(1_1)$, which does not take value 1 on all non-trivial 5-elements. Second $e_0 \cdot \mathrm{Ind}_N^G(1_3) = \chi_{14} + \chi_{15} + \chi_{16} + \chi_{17}$. Again by the degree criteria, we must have that $\Gamma_G(1_3)$ affords $e_0 \cdot \mathrm{Ind}_N^G(1_3)$, as it takes value 1 on all the non-trivial 5-elements. By Theorem 2.11, $\Gamma_G(1_3)$ is endotrivial whereas $\Gamma_G(1_1)$ is not. Whence $TT(G) \cong \mathbb{Z}/2$.

6.5. **The group** $G = \mathrm{M}_{23}$ **in characteristic** 3. In this case P is elementary abelian of order 9 and $N \cong 3^2 : Q_8.2$. It follows that $X(N) \cong (\mathbb{Z}/2)^2$.

By [LMS13, Thm. 7.1], M_{23} has a simple self-dual endotrivial module S_{253} of dimension 253. Hence $\mathbb{Z}/2 \cong \langle [S_{253}] \rangle \leq TT(G)$.

Now, the natural permutation kG-module on 23 points has a 22-dimensional composition factor S_{22}, which must be a direct summand. Therefore S_{22} is a trivial source module with vertex P and its kN-Green correspondent is simple (see [DK09, §3.7]). Thus it must have dimension at most 2. Since $22 \equiv 1 \pmod 3$, we conclude that S_{22} is the kG-Green correspondent of a 1-dimensional kN-module. But as $22 \not\equiv 1 \pmod{|P|}$, the module S_{22} is not endotrivial. Consequently $TT(G) \cong \mathbb{Z}/2$.

6.6. **The group** $G = {}^2\mathrm{F}_4(2)'$.

Characteristic 3. In this case P is extraspecial of order 27 and exponent 3 and $N \cong 3_+^{1+2} : D_8$. Hence $X(N) \cong (\mathbb{Z}/2)^2$ and we denote $1_N, 1_1, 1_2, 1_3$ the four non-isomorphic 1-dimensional kN-modules.

The group G has a maximal subgroup $H \cong \mathrm{L}_3(3).2$ of index 1600 which contains N and $X(H) = \{1_H, 1_a\}$ has order 2. We may assume that $\Gamma_H(1_N) = 1_H$ and $\Gamma_H(1_1) = 1_a$. Induction from N to H yields

$$\mathrm{Ind}_N^H(1_2) = \chi_5 + \chi_{13} \quad \text{and} \quad \mathrm{Ind}_N^H(1_3) = \chi_6 + \chi_{14}.$$

Since $\chi_5(1) = \chi_6(1) = 13$ and $\chi_{13}(1) = \chi_{14}(1) = 39$, and $13, 39, 13+39 \not\equiv 1 \pmod{|P|}$, the kH-Green correspondents $\Gamma_H(1_2)$ and $\Gamma_H(1_3)$ cannot be endotrivial. Similarly, $\mathrm{Ind}_H^G(1_a) = \chi_7 + \chi_{19}$ where $\chi_7(1) = 300 \not\equiv 1 \pmod{|P|}$, $\chi_{19}(1) = 1300 \not\equiv 1 \pmod{|P|}$ and $(\chi_7 + \chi_{19})(1) \not\equiv 1 \pmod{|P|}$ so that the kG-Green correspondent of 1_1 is not endotrivial. Therefore $TT(H) \cong \mathbb{Z}/2$ and $TT(G) \cong \{[k]\}$.

Characteristic 5. In this case P is elementary abelian of rank 2 and $N \cong (5^2) : (4.\mathfrak{A}_4)$. Theorem 3.3 says that P is a T.I. set, so that by Lemma 2.8(4),(5) we get that $TT(G) \cong TT(N) \cong X(N)$ because N is strongly p-embedded in G. Finally we compute that $X(N) \cong \mathbb{Z}/6$.

6.7. **The group $G = \mathrm{HS}$.**

Characteristic 3. In this case $P \cong 3^2$ and $N \cong 2 \times (3^2.SD_{16})$. So $X(N) \cong (\mathbb{Z}/2)^3$. By [LMS13, Thm. 7.1], G has three simple endotrivial modules S_4, S_5, S_6 all of dimension 154, and affording the characters χ_4, χ_5, χ_6 respectively. From [LM15, Sec.2] we have $TT(G) = \langle [S_4], [S_5], [S_6] \rangle \cong (\mathbb{Z}/2)^2$. Alternatively, a MAGMA computation (not involving Balmer's method described above) shows that there exists a kN-module $1_a \in X(N)$ such that its kG-Green correspondent is of dimension $175 \not\equiv 1 \pmod 9$, so that $TT(G)$ certainly cannot have order 8.

Characteristic 5. In this case $P \cong 5^{1+2}_+$, and $N \cong 5^{1+2}_+ : (8.2)$. Also G has a maximal subgroup $H^* \cong \mathrm{U}_3(5).2$ containing N. We get $X(N) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$, $X(H^*) \cong \mathbb{Z}/2$. We note that N is strongly 5-embedded in H^*, so that $TT(H^*) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$ by Lemma 2.8. Now if $1_a \in X(H^*)$ is non-trivial, we have $e_0 \cdot \mathrm{Ind}_{H^*}^G(1_a) = \chi_2 + \chi_5$, where $\chi_2(1) = 22, \chi_5(1) = 154$. Therefore $\Gamma_G(1_1) = e_0 \cdot \mathrm{Ind}_{H^*}^G(1_a)$ and affords $\chi_2 + \chi_5$, but is not endotrivial since its dimension is $176 \not\equiv 1 \pmod{|P|}$. From the group structure of $X(N)$, we see that $TT(G) \leq \mathbb{Z}/4$. Now the linear character $1_5 \in X(N)$ has order 4 and

$$e_0 \cdot \mathrm{Ind}_N^G(1_5) = \chi_8 + \chi_9 + \chi_{10} + \chi_{16} + \chi_{17} + 2 \cdot \chi_{22}.$$

(Note that B_0 is the unique 5-block with full defect.) It follows from the possible values of trivial source modules given by Lemma 2.12 and the 5-decomposition matrix of G that $\Gamma_G(1_5)$ affords either the character $\chi_8 + \chi_{10}$ or the character $\chi_8 + \chi_{22}$. In both cases we obtain that $\Gamma_G(1_5)$ is endotrivial by Theorem 2.11 and we conclude that $TT(G) \cong \mathbb{Z}/4$.

6.8. **The group $G = \mathrm{J}_3$ in characteristic 3.** In this case P has order 3^5 and $N \cong 3^2.(3^{1+2}) : 8$ so that $X(N) \cong \mathbb{Z}/8$ (note that N is maximal in G). The linear character 1_4 has order 4 in $X(N)$ and

$$e_0 \cdot \mathrm{Ind}_N^G(1_4) = 2\chi_{10} + 2\chi_{13} + \chi_{14} + \chi_{15} + \chi_{16} + \chi_{17} + \chi_{18} + \chi_{19}.$$

It follows from the character table of J_3 that if $\dim_k \Gamma_G(1_4) \equiv 1 \pmod{|P|}$, then its character cannot take value 1 on all non-trivial 3-elements. Hence, by Theorem 2.11, $\Gamma_G(1_4)$ is not endotrivial. This proves that $TT(G) \leq \mathbb{Z}/2$.

6.9. **The group $G = \mathrm{M}_{24}$ in characteristic** 3. We show that $TT(G) = \{[k]\}$.

In this case P is extraspecial of order 27 and exponent 3 and $N \cong 3^{1+2}_+ : D_8$ where D_8 is a dihedral group of order 8. From [CCN+85], we see that G has a maximal subgroup $H \cong M_{12} : 2$ containing N. Therefore $X(H) \cong \mathbb{Z}/2$ and $X(N) \cong (\mathbb{Z}/2)^2$. We denote by $1_H, 1_a$ the elements of $X(H)$ and by $1_N, 1_1, 1_2$ and 1_3 the elements of $X(N)$. By Green correspondence, we may assume that $\Gamma_H(1_N) = 1_H$ and $\Gamma_H(1_1) = 1_a$. Inducing the character 1_3 to H gives

$$e_0 \cdot \mathrm{Ind}_N^H(1_3) = \chi_{12} + \chi_{15} + \chi_{16} + \chi_{17} + \chi_{21}.$$

(Note that the principal block is the unique 3-block of kH with full defect). Using Theorem 2.11 and Lemma 2.12 it is easy to check that for degree reasons, or character values on 3-elements, $\Gamma_H(1_3)$ cannot be endotrivial. Consequently $TT(H) = X(H) \cong \mathbb{Z}/2$.

Finally $\mathrm{Ind}_H^G(1_a) = \chi_2 + \chi_{17}$, where $\chi_2(1) = 23$ and $\chi_{17}(1) = 1265 \equiv 23 \pmod{|P|}$, so that again for degree reasons the kG-Green correspondent of 1_a is not endotrivial either. The claim follows.

6.10. The group $G = \mathrm{McL}$ and its 3-fold cover.

Characteristic 3. In this case P has order 3^6 and $N \cong 3^4 : (3^2 : Q_8)$ so that $X(N) \cong (\mathbb{Z}/2)^2$. In addition G has a subgroup $H \cong 3^{1+4} : (2.\mathfrak{S}_5)$. Since $|H : N| = 10 < |P|$, we have $TT(H) \cong X(H) \cong \mathbb{Z}/2$. So let 1_a be the non-trivial linear character of H. Then $\mathrm{Ind}_H^G(1_a) = \chi_3 + \chi_{15} + \chi_{20}$. Now as $\chi_3, \chi_{15}, \chi_{20}$ all take a value strictly greater than 1 on conjugacy class $3a$, $\Gamma_G(1_a)$ cannot be endotrivial by Theorem 2.11. Whence $TT(G) = \{[k]\}$, whereas the group $TT(3.\mathrm{McL})$ is trivial by Lemma 3.1(2).

Characteristic 5. In this case $P = 5_+^{1+2}$ and $N \cong 5_+^{1+2} : 3 : 8$ is a maximal subgroup of G. Moreover P is a T.I. set by Theorem 3.3 and therefore by Lemma 2.8(4),(5) we get that $TT(G) \cong X(N) \cong \mathbb{Z}/8$. The situation is similar for the 3-fold cover 3.McL, i.e. $\widetilde{P} \cong 5_+^{1+2}$ and is a T.I. set as well. The normaliser $\widetilde{N}$ is an extension of degree 3 of N and $X(\widetilde{N}) \cong \mathbb{Z}/24$. Whence $TT(3.\mathrm{McL}) \cong X(\widetilde{N}) \cong \mathbb{Z}/24$.

6.11. The group $G = \mathrm{He}$.

Characteristic 3. In this case $P \cong 3_+^{1+2}$ and $N \cong 3_+^{1+2} : D_8$ so that $X(N) \cong (\mathbb{Z}/2)^2$. The group G has a maximal subgroup $H \cong 3.A_7.2$ containing N. It follows from Lemma 2.8(3) that $TT(H) \cong X(H)$, where $X(H) = \{1_H, 1_2\} \cong \mathbb{Z}/2$. Hence $TT(G) \leq TT(H) \cong \mathbb{Z}/2$ via restriction and we only need to check whether $\Gamma_G(1_2)$ is endotrivial. But $\mathrm{Ind}_H^G(1_2) = \chi_{14} + \chi_{15} + \chi_{22} + 2 \cdot \chi_{29} + \chi_{30} + \chi_{31}$ so that it follows easily from the values of $\chi_{14}, \chi_{15}, \chi_{22}, \chi_{29}, \chi_{30}, \chi_{31} \in \mathrm{Irr}(G)$ that the character afforded by $\Gamma_G(1_2)$ cannot take value 1 on conjugacy class $3b$. Whence $TT(G) \cong \{[k]\}$.

Characteristic 5. In this case P is elementary abelian of order 25 and $N \cong 5^2 : (4.\mathfrak{S}_4)$. So $X(N) \cong \mathbb{Z}/6$. By [LMS13, Thm. 7.1], G has two simple trivial source endotrivial modules S_{51} and S_{51}^*, both of dimension 51, affording the characters $\chi_2, \chi_3 \in \mathrm{Irr}(G)$. Now $\chi_1 \downarrow_N$ and $\chi_2 \downarrow_N$ have exactly two linear constituents χ_4 and χ_6, both of which have order 3 in $X(N)$. Therefore S_{51}, S_{51}^* have order 3 in $TT(G)$, since they are the kG-Green correspondents of 1-dimensional modules of order 3 in $X(N)$, showing that $TT(G)$ has order at least 3. Finally a MAGMA computation shows that $TT(G) \cong \mathbb{Z}/3$.

6.12. The group $G = \mathrm{Ru}$ and its cover in characteristic 3. In this case P is extraspecial of order 27 and exponent 3 and $N \cong 3_+^{1+2} : SD(16)$. So $X(N) \cong (\mathbb{Z}/2)^2$. For $2.G$, we have $\widetilde{P} \cong P$ and $X(\widetilde{N}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$.

By [LMS13, Thm. 7.1], G has a self-dual simple trivial source endotrivial module S_{406} of dimension 406, affording the character $\chi_4 \in \mathrm{Irr}(G)$. Therefore $\mathbb{Z}/2 \cong \langle [S_{406}] \rangle \leq TT(G)$. Again by [LMS13, Thm. 7.1], $2.G$ has two faithful simple endotrivial modules S_{28} and S_{28}^*, both of dimension 28, affording the characters $\chi_{37}, \chi_{38} \in \mathrm{Irr}(2.G)$. Moreover $\chi_{37} \otimes \chi_{37} = \chi_2 + \chi_4$ where χ_2 has defect zero. Reduction modulo 3 yields $S_{28} \otimes S_{28} \cong S_{406} \oplus (\mathrm{proj})$ saying that S_{28} is a trivial source module of order 4 in $TT(\widetilde{G})$. Therefore $\mathbb{Z}/4 \cong \langle [S_{28}] \rangle \leq TT(2.G)$.

Now G has a maximal subgroup $H \cong (2^6 : \mathrm{U}_3(3)) : 2$ containing N with $X(H) \cong \mathbb{Z}/2$. If $1_2 \in X(H)$ denotes the non-trivial character of H, then $e_0 \cdot \mathrm{Ind}_H^G(1_2) = \chi_9 + \chi_{25}$. For degree reasons $\Gamma_G(1_1)$ affords $\chi_9 + \chi_{25} \in \mathbb{Z}\,\mathrm{Irr}(G)$, but $\chi_9 + \chi_{25}$ takes value 31 on class $3a$, so that $\Gamma_G(1_1)$ is not endotrivial by Theorem 2.11. This yields $TT(H) \cong TT(G) \cong \mathbb{Z}/2$ and $TT(2.G) \cong \mathbb{Z}/4$.

6.13. **The group $G = $ Suz.**

Characteristic 3. In this case $P \cong 3^5 : 3^2$, $N \cong 3^5 : (3^2 : SD_{16})$ and $X(N) \cong (\mathbb{Z}/2)^2$. Moreover G has a subgroup $H \cong 3^5 : M_{11}$ containing N, so that $TT(G) \leq TT(H)$ via restriction. Inducing the three non-trivial linear characters $1_1, 1_2, 1_4$ of N to H we get:

$$e_0 \cdot \text{Ind}_N^H(1_1) = \chi_2 + \chi_9, \quad e_0 \cdot \text{Ind}_N^H(1_2) = \chi_5 + \chi_8, \quad e_0 \cdot \text{Ind}_N^H(1_4) = \chi_{10}$$

(Note that H has a unique block of full defect.) Since $\chi_2, \chi_5, \chi_8, \chi_9, \chi_{10} \in \text{Irr}(H)$ all have degree strictly greater than 1 and class $3a$ in their kernel, it follows that the kH-Green correspondents $\Gamma_H(1_1), \Gamma_H(1_2), \Gamma_H(1_4)$ cannot be endotrivial by Theorem 2.11. A fortiori $TT(G) = TT(H) = \{[k]\}$.

Characteristic 5. By [LMS13, Thm. 7.1] the character $\chi_5 \in \text{Irr(G)}$ reduces modulo 5 to a self-dual endotrivial kG-module S_{1001} of dimension 1001. Therefore $\mathbb{Z}/2 \leq TT(\text{Suz}) \leq \mathbb{Z}/2 \oplus \mathbb{Z}/4$ since $X(N) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$. Moreover, G has a maximal subgroup $H \cong J_2 : 2$ containing the normaliser $N \cong 5^2 : (4 \times \mathfrak{S}_3)$. Similar computations as above show that H has two trivial source endotrivial modules of order 4, both affording the character $\chi_{18} + \chi_{19}$, with $\chi_{18}, \chi_{19} \in \text{Irr}(H)$. The group H also has a trivial source module of order 2 affording the character $\chi_{18} + \chi_{23}$ of degree 666, hence not endotrivial. It follows that $\mathbb{Z}/2 \leq TT(G) \leq \mathbb{Z}/4$. Finally a MAGMA computation shows that $TT(G) \cong \mathbb{Z}/2$.

6.14. **The group $G = $ O'N and its three-fold cover in characteristic 7.** In this case $P = 7_+^{1+2}$ and $N \cong 7_+^{1+2} : (D_8 \times 3)$ so that $X(N) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/6$. The principal block is the unique block with full defect. Now N is contained in a maximal subgroup $H \cong L_3(7) : 2$. First, let $1_a \in X(H) \cong \mathbb{Z}/2$ be the non-trivial linear character of H. Then $e_0 \cdot \text{Ind}_H^G(1_a) = \chi_{10}$, which does not take value 1 on 7-elements. Hence $\Gamma_G(1_a)$ is not endotrivial by Theorem 2.11. Since $TT(G)$ identifies with a subgroup of $TT(H)$ via restriction, it remains to show that $TT(H) = X(H)$. For this we let $1_i \in X(N)$, $1 \leq i \leq 12$ denote the non-trivial linear characters of N. The character 1_6 has order 3 and $\text{Ind}_N^H(1_6) = \chi_7 + \chi_9$. The characters $1_2, 1_4$ have order 2, the kH-Green correspondent of 1_4 is 1_a and $\text{Ind}_N^H(1_2) = \chi_6 + \chi_{19}$. Therefore the characters of $\Gamma_H(1_6)$ and $\Gamma_H(1_2)$ cannot take value 1 on non-trivial 7-elements, so that $\Gamma_H(1_6)$ and $\Gamma_H(1_2)$ are not endotrivial by Theorem 2.11. It follows that $TT(H) = X(H) \cong \mathbb{Z}/2$ and $TT(G) \cong \{[k]\}$. Similarly to the situation for G, the three-fold cover $3.G = 3.$O'N has a maximal subgroup of shape $3 \times H$ containing $\widetilde{N}$. We have $X(\widetilde{N}) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/6$ and $X(3 \times H) \cong \mathbb{Z}/6$. Again since Green and Brauer correspondences commute, we only need to consider faithful modules. Let $1_c, 1_d \in X(3 \times H)$ be the two modules of order 6. Inducing $1_c, 1_d$ to $3.G$ in GAP, we get that the characters afforded by $\Gamma_G(1_c)$ and $\Gamma_G(1_d)$ are $\chi_{59}, \chi_{60} \in \text{Irr}(3.G)$. But both these characters take the value 15 on conjugacy class $7a$, so that $\Gamma_G(1_c), \Gamma_G(1_d)$ cannot be endotrivial by Theorem 2.11. It follows that $TT(3.G)$ has no faithful element of order 3 and so $TT(3.G) \cong \{[k]\}$.

6.15. **The group $G = $ Co$_3$ in characteristic 3.** In this case $|P| = 3^7$. We choose maximal subgroups $H_1 \cong 3^5 : (2 \times M_{11})$ and $H_2 \cong 3^{1+4} : (4.\mathfrak{S}_6)$ of G both containing N so that $TT(G) \leq TT(H_1)$ and $TT(G) \leq TT(H_2)$ via restriction. Since $|H_1 : N| = 55$ and $|H_2 : N| = 10$, trivial source endotrivial kH_1-, kH_2-modules must be 1-dimensional by Lemma 2.2(7), that is $TT(H_1) \cong X(H_1) \cong \mathbb{Z}/2$ and $TT(H_2) \cong X(H_2) \cong (\mathbb{Z}/2)^2$. Then inducing the linear characters of N in GAP, we find that there is a module $1_a \in X(N)$

such that $\Gamma_{H_1}(1_a) \in X(H_1)$ has order 2, but $\Gamma_{H_2}(1_a)$ has dimension 10 and thus cannot be endotrivial. Whence $\Gamma_G(1_a)$ is not endotrivial either and $TT(G) = \{[k]\}$.

6.16. The group $G = \mathrm{Co}_2$ in characteristic 3. In this case $|P| = 3^6$. We choose maximal subgroups $H_1 \cong 3^5 : (2 \times \mathrm{M}_{11})$ and $H_2 \cong 3^{1+4} : (4.\mathfrak{S}_6)$ of G both containing N so that $TT(G) \leq TT(H_1)$ and $TT(G) \leq TT(H_2)$ via restriction. Since $|H_2 : N| = 40$, trivial source endotrivial kH_2-modules must be 1-dimensional by Lemma 2.2(7) so that $TT(H_2) \cong X(H_2) \cong \mathbb{Z}/2$. Let $1_a \in X(N)$ be the module such that $\Gamma_{H_2} \in X(H_2)$ is 1-dimensional of order 2. Then

$$\mathrm{Ind}_N^{H_1}(1_a) = \chi_{11} + \chi_{30} + \chi_{31} + \chi_{37}$$

where $\chi_{11}, \chi_{30}, \chi_{31}, \chi_{37} \in \mathrm{Irr}(H_1)$ all take values greater than 1 on class $3b$. Thus, by Theorem 2.11, $\Gamma_{H_1}(1_a)$ cannot be endotrivial, and neither can $\Gamma_G(1_a)$. Hence $TT(G) = \{[k]\}$.

6.17. The group $G = \mathrm{Fi}_{22}$ and its covers.

Characteristic 3. In this case P has order 3^9, exponent 9 and is an extension of an elementary abelian 3-group of order 3^5 with a Sylow 3-subgroup of $\mathrm{U}_4(2)$, which is itself a wreath product $3 \wr 3$. Then N is contained in a subgroup $H \cong \mathrm{O}_7(3)$ of G. By [CMN06, Theorem 6.2], we have $TT(H) = \{[k]\}$, and therefore $TT(G) = \{[k]\}$.

Characteristic 5. In this case $P \cong 5^2$, N has order $2^5 \cdot 3 \cdot 5^2$ and $X(N) \cong \mathbb{Z}/4$. By [LMS13, Thm. 7.1], the character $\chi_4 \in \mathrm{Irr}(G)$ reduces modulo 5 to a self-dual endotrivial kG-module S_{1001} of dimension 1001, and by [LM15, Sec.2] we have $TT(G) = \langle [S_{1001}] \rangle \cong \mathbb{Z}/2$. Furthermore [LM15, Sec.2] also shows that $TT(2.G) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, while $X(\widetilde{N}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$, and $TT(3.G) \cong \mathbb{Z}/6$, where by [LMS13, Thm. 7.1], the characters $\chi_{115}, \overline{\chi_{115}} \in \mathrm{Irr}(3.\mathrm{Fi}_{22})$ reduce modulo 5 to $k[3.G]$-modules S_{351} and $(S_{351})^*$, both faithful simple trivial source endotrivial of dimension 351. It follows that $TT(6.G) \cong \mathbb{Z}/6 \oplus \mathbb{Z}/2$.

6.18. The group $G = \mathrm{HN}$.

Characteristic 3. In this case $|P| = 3^6$ and we can choose maximal subgroups $H_1 \cong 3^4 : 2.(\mathfrak{A}_4 \times \mathfrak{A}_4).4$ and $H_2 \cong 3^{1+4} : (4.\mathfrak{A}_5)$ of G, both containing N so that $TT(G) \leq TT(H_1)$ and $TT(G) \leq TT(H_2)$ via restriction. Since $|H_1 : N| = 16$ and $|H_2 : N| = 10$, trivial source endotrivial kH_1- and kH_2-modules must be 1-dimensional by Lemma 2.2(7), that is $TT(G) \leq TT(H_1) \cong X(H_1) \cong \mathbb{Z}/4$ and $TT(G) \leq TT(H_2) \cong X(H_2) \cong \mathbb{Z}/2$. But if $1_a \in X(N)$ is of order 2 such that $\Gamma_{H_2}(1_a)$ is 1-dimensional, then $\Gamma_{H_1}(1_a)$ affords the irreducible character $\chi_{19} \in \mathrm{Irr}(H_1)$ of degree 16, hence is not endotrivial and neither is $\Gamma_G(1_a)$, whence $TT(G) = \{[k]\}$.

Characteristic 5. In this case $|P| = 5^6$. Similarly as above, we choose maximal subgroups $H_1 \cong 5^{1+4} : 2^{1+4}.5.4$ and $H_2 \cong 5^2.5.5^2.(4.\mathfrak{A}_5)$ of G, both containing N. Since $|H_1 : N| = 16$ and $|H_2 : N| = 6$, we get $TT(H_1) \cong X(H_1) \cong \mathbb{Z}/4$ and $TT(H_2) \cong X(H_2) \cong \mathbb{Z}/2$. Now if $1_a \in X(N)$ denotes the 1-dimensional module such that $\Gamma_{H_1}(1_a) \in X(H_1)$ has order 2, then inducing 1_a to H_2 at the level of characters yields $\mathrm{Ind}_N^{H_2}(1_a) = \chi_7 + \chi_8$ where both $\chi_7, \chi_8 \in \mathrm{Irr}(H_2)$ have degree 3. Therefore $\Gamma_{H_2}(1_a) = \mathrm{Ind}_N^{H_2}(1_a)$ with dimension 6 and is not endotrivial, whence $TT(G) = \{[k]\}$.

6.19. The group $G = \mathrm{Ly}$.

Characteristic 3. In this case $|P| = 3^7$. We choose maximal subgroups $H_1 \cong 3^5 : (2 \times \mathrm{M}_{11})$ and $H_2 \cong 3^{2+4} : 2.\mathfrak{A}_5.D_8$ both containing N. Since $|H_1 : N| = 55$ and $|H_2 : N| = 10$, we get $TT(G) \leq TT(H_1) \cong X(H_1) \cong \mathbb{Z}/4$ and $TT(G) \leq TT(H_2) \cong X(H_2) \cong \mathbb{Z}/2$. Thus an argument similar to that used for the simple group HN applies and it follows that $TT(G) = \{[k]\}$.

Characteristic 5. In this case G contains a maximal subgroup $H \cong \mathrm{G}_2(5)$, so that $TT(G)$ identifies with a subgroup of $TT(H)$ via restriction. Since $TT(H) = \{[k]\}$ by [CMN06, Thm. A], we have $TT(G) = \{[k]\}$.

6.20. The group $G = \mathrm{Fi}_{23}$ in characteristic 5. In this case P is elementary abelian of rank 2 and $X(N) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$. By [LMS13, Thm. 7.1] the character $\chi_9 \in \mathrm{Irr}(G)$ reduces modulo 5 to a self-dual endotrivial kG-module S_{111826} of dimension 111826. Therefore $\mathbb{Z}/2 \leq TT(G) \leq \mathbb{Z}/2 \oplus \mathbb{Z}/4$. Now a maximal subgroup of G is $H = 2.\mathrm{Fi}_{22}$ and H contains N, so that $TT(G) \leq TT(2.\mathrm{Fi}_{22}) \cong (\mathbb{Z}/2)^2$ via restriction. Moreover, if S_{1001} denotes the self-dual simple endotrivial module of Subsection 6.17 of dimension 1001, we compute that $\Gamma_G(S_{1001})$ affords $\chi_5 \in \mathrm{Irr}(G)$ with degree $25806 \not\equiv 1 \pmod{|P|}$. It follows that $TT(G) \cong \mathbb{Z}/2$.

6.21. The group $G = \mathrm{Co}_1$.

Characteristic 3. We choose maximal subgroups $H_1 \cong 3^{1+4}.2.\mathrm{U}_4(2).2$ and $H_2 \cong 3^6 : 2.\mathrm{M}_{12}$ both containing N and of index smaller than $|P| = 3^9$. Then an argument similar to that used for the simple group HN applies and it follows that $TT(G) = \{[k]\}$.

Characteristic 5. In this case $|P| = 5^4$ and $X(N) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4$. Moreover the group G has a maximal subgroup $H \cong 5^{1+2} : \mathrm{GL}_2(5)$ containing N and such that $|H : N| = 5$. Thus by Lemma 2.2(7), $TT(G) \leq TT(H) \cong X(H) \cong \mathbb{Z}/4$.

6.22. The group $G = \mathrm{J}_4$.

Characteristic 3. In this case $P = 3_+^{1+2}$, $N = (2 \times 3_+^{1+2} : 8) : 2$ and $X(N) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$. We show that $TT(G)$ has no element of order 2, hence is trivial. The group G has a maximal subgroup $H \cong 2_+^{1+12}.3.M_{22} : 2$ containing N and such that $X(H) \cong \mathbb{Z}/2$. Let $1_1, 1_2, 1_3$ be the three linear characters of N of order 2. Then $\Gamma_H(1_2) = 1_a$ the non-trivial linear character of H. The irreducible constituents of $\mathrm{Ind}_N^H(1_1)$ and $\mathrm{Ind}_N^H(1_3)$ all take a value greater than 1 on class $3a$, so that by Theorem 2.11 $\Gamma_H(1_1)$ and $\Gamma_H(1_3)$ are not endotrivial. Finally $e_0 \cdot \mathrm{Ind}_H^G(1_a) = \chi_{40} + \chi_{52}$, where both $\chi_{40}, \chi_{52} \in \mathrm{Irr}(G)$ take value greater than 1 on conjugacy class $3a$. Hence $\Gamma_G(1_2)$ is not endotrivial either. The claim follows.

Characteristic 11. The group J_4 possesses a T.I. Sylow 11-subgroup $P = 11_+^{1+2}$ by Theorem 3.3, and we know that $N \cong 11_+^{1+2} : (5 \times (2.\mathfrak{S}_4))$ (cf [CCN+85]). Therefore $X(N) \cong \mathbb{Z}/10$ and it follows from Lemma 2.8(4),(5) that $TT(G) \cong X(N) \cong \mathbb{Z}/10$.

6.23. The group $G = \mathrm{B}$ in characteristic 5. The group B has a Sylow 5-subgroup of order 5^6, whose normaliser N is contained in a maximal subgroup $H \cong \mathrm{HN}{:}2$. It follows directly from Subsection 6.18 that $TT(H) \cong \mathbb{Z}/2$, hence $TT(G) \leq \mathbb{Z}/2$. (We note that in this case $X(N) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4$.)

7. Endotrivial modules for sporadic groups - summary of results

Table 3 below summarises the isomorphism types of the groups $T(G)$ of endotrivial kG-modules that we could determine in the case when the Sylow p-subgroups have p-rank at least 2. An empty entry means that the Sylow p-subgroups for the corresponding prime are either cyclic or $p \nmid |G|$. An isomorphism "$\cong T(G)$" indicates that the isomorphism is obtained via inflation. A question mark indicates that only a partial result for the structure of $TT(G)$ has been obtained. In these cases we summarise the structure of the group $X(N)$ in Table 4, which we obtain using [GL83, GLS98, Bre12, BCP97]. The motivation for such computations is Lemma 2.8(2), asserting that $TT(G) \leq X(N)$. A question mark bounded by a group, e.g. $(? \leq \mathbb{Z}/2)$ for the group B in characteristic 5, indicates that we have found a sharper bound for $TT(G)$ than the one given by $X(N)$.

Finally, we assume that G is quasi-simple with $G/Z(G)$ sporadic simple and a Sylow p-subgroup P of G is cyclic. Then $P \cong C_p$ and $p \neq 2$ (see [CCN+85]). We use Theorem 2.9 with, in our case, $H = N_G(P)$ and $e = |N_G(P) : C_G(P)|$ is the inertial index of the principal block of G. The results come down to a routine computation using the structure of H which can easily be computed using [GLS98, GAP13], and from which we get $X(H)$ and hence $T(G)$. We present them in Table 5.

To compute $T(G)$ in this case, we recall that $T(G) \cong T(H)$, and the short exact sequence

$$0 \longrightarrow X(H) \longrightarrow T(H) \xrightarrow{\mathrm{Res}_P^H} T(P) \longrightarrow 0$$

splits if and only if the class $[\Omega^2(k_H)]$ is a square in $X(H)$. In particular the sequence splits if e is odd in which case $T(G) \cong X(H) \oplus \mathbb{Z}/2$. Moreover $T(G)$ always contains a subgroup $\langle \Omega \rangle \cong \mathbb{Z}/2e$. If $|X(H)| = e$, then the principal block is the unique block containing endotrivial modules so that $T(G) = \langle \Omega \rangle \cong \mathbb{Z}/2e$. If e is even, we use the distribution of endotrivial modules into blocks as described in [LMS13, Table 7]. Moreover, if $m.G$ is a covering group of G and $p \nmid m$, then $T(G)$ identifies with a subgroup of $T(m.G)$ via inflation. This is enough to conclude in all cases.

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TABLE 3. $T(G)$ for covering groups of sporadic groups with p^2 dividing $|G|$

Group G	$p=2$	$p=3$	$p=5$	$p=7$	$p=11$	$p=13$
M_{11}	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$				
M_{12}	$\mathbb{Z}$	$\mathbb{Z}^3$				
$2.M_{12}$	$\mathbb{Z}$	$\cong T(M_{12})$				
J_1	$\mathbb{Z}$					
M_{22}	$\mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$				
$2.M_{22}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$				
$3.M_{22}$	$\mathbb{Z}$	$\mathbb{Z}$				
$4.M_{22}$	$\mathbb{Z}$	$\cong T(2.M_{22})$				
$6.M_{22}$	$\mathbb{Z}$	$\mathbb{Z}$				
$12.M_{22}$	$\mathbb{Z}$	$\mathbb{Z}$				
J_2	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$			
$2.J_2$	$\mathbb{Z}$	$\cong T(J_2)$	$\cong T(J_2)$			
M_{23}	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2$				
$^2F_4(2)'$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}/6$			
HS	$\mathbb{Z}$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}/4$			
$2.HS$	$\mathbb{Z}$	$\cong T(HS)$	$\cong T(HS)$			
J_3	$\mathbb{Z}$	$\mathbb{Z} \oplus (? \leq \mathbb{Z}/2)$				
$3.J_3$	$\mathbb{Z} \oplus \mathbb{Z}/3$	$\mathbb{Z}$				
M_{24}	$\mathbb{Z}$	$\mathbb{Z}^2$				
McL	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/8$			
$3.McL$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/24$			
He	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}/3$	$\mathbb{Z}^3 \oplus ?$		
Ru	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}^2 \oplus ?$			
$2.Ru$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/4$	$\cong T(Ru)$			
Suz	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2$			
$2.Suz$	$\mathbb{Z}$	$\cong T(Suz)$	$\cong T(Suz)$			
$3.Suz$	$\mathbb{Z}$	$\mathbb{Z}$	$\cong T(Suz)$			
$6.Suz$	$\mathbb{Z}$	$\mathbb{Z}$	$\cong T(Suz)$			
O'N	$\mathbb{Z}$	$\mathbb{Z} \oplus (? \leq \mathbb{Z}/2)$		$\mathbb{Z}^3$		
$3.O'N$	$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}^3$		
Co_3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus ?$			
Co_2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus ?$			
Fi_{22}	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2$			
$2.Fi_{22}$	$\mathbb{Z}$	$\cong T(Fi_{22})$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$			
$3.Fi_{22}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/6$			
$6.Fi_{22}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/2$			
$F_5 = HN$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$			
Ly	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$			
Th	$\mathbb{Z}$	$\mathbb{Z} \oplus ?$	$\mathbb{Z} \oplus ?$	$\mathbb{Z} \oplus ?$		
Fi_{23}	$\mathbb{Z}$	$\mathbb{Z} \oplus ?$	$\mathbb{Z} \oplus \mathbb{Z}/2$			
Co_1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2 \oplus (? \leq \mathbb{Z}/4)$	$\mathbb{Z} \oplus ?$		
$2.Co_1$	$\mathbb{Z}$	$\cong T(Co_1)$	$\cong T(Co_1)$	$\cong T(Co_1)$		
J_4	$\mathbb{Z}$	$\mathbb{Z}$			$\mathbb{Z} \oplus \mathbb{Z}/10$	
Fi_{24}'	$\mathbb{Z}$	$\mathbb{Z} \oplus ?$	$\mathbb{Z} \oplus ?$	$\mathbb{Z}^3 \oplus ?$		
$3.Fi_{24}'$	$\mathbb{Z}$	$\mathbb{Z}$	$\cong T(Fi_{24}')$	$\cong T(Fi_{24}')$		
$F_2 = B$	$\mathbb{Z}$	$\mathbb{Z} \oplus ?$	$\mathbb{Z} \oplus (? \leq \mathbb{Z}/2)$	$\mathbb{Z} \oplus ?$		
$2.F_2 = 2.B$	$\mathbb{Z}$	$\cong T(B)$	$\cong T(B)$	$\cong T(B)$		
$F_1 = M$	$\mathbb{Z}$	$\mathbb{Z} \oplus ?$	$\mathbb{Z} \oplus ?$	$\mathbb{Z} \oplus ?$	$\mathbb{Z} \oplus ?$	$\mathbb{Z}^2 \oplus ?$

TABLE 4. The structure of $X(N)$ for the missing cases in Table 3

G	p	$X(N)$	G	p	$X(N)$	G	p	$X(N)$
He	7	$\mathbb{Z}/6$	Th	7	$\mathbb{Z}/6$	B	7	$\mathbb{Z}/2 \oplus \mathbb{Z}/6$
Ru	5	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	Fi$_{23}$	3	$(\mathbb{Z}/2)^3$	M	3	$(\mathbb{Z}/2)^3$
O'N	3	$\mathbb{Z}/2$	Co$_1$	7	$\mathbb{Z}/3 \oplus \mathbb{Z}/3$	M	5	$\mathbb{Z}/2 \oplus (\mathbb{Z}/4)^2$
Co$_3$	5	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	Fi$'_{24}$	3	$(\mathbb{Z}/2)^3$	M	7	$\mathbb{Z}/2 \oplus \mathbb{Z}/6$
Co$_2$	5	$\mathbb{Z}/4$	Fi$'_{24}$	5	$\mathbb{Z}/4$	M	11	$\mathbb{Z}/5$
Th	3	$(\mathbb{Z}/2)^2$	Fi$'_{24}$	7	$\mathbb{Z}/2 \oplus \mathbb{Z}/6$	M	13	$\mathbb{Z}/12$
Th	5	$\mathbb{Z}/4$	B	3	$(\mathbb{Z}/2)^3$			

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TABLE 5. $T(G)$ for covering groups of sporadic groups with cyclic Sylow p-subgroups

G	p	$X(H)$	e	$T(G)$	G	p	$X(H)$	e	$T(G)$
$^2F_4(2)'$	13	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$	He	17	$\mathbb{Z}/8$	8	$\mathbb{Z}/16$
M_{11}	5	$\mathbb{Z}/4$	4	$\mathbb{Z}/8$	Ru	7	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$
M_{11}	11	$\mathbb{Z}/5$	5	$\mathbb{Z}/10$	2.Ru	7	$\mathbb{Z}/6$	6	$\cong T(\mathrm{Ru})$
M_{12}	5	$\mathbb{Z}/2\oplus\mathbb{Z}/4$	4	$\mathbb{Z}/2\oplus\mathbb{Z}/8$	Ru	13	$\mathbb{Z}/12$	12	$\mathbb{Z}/24$
$2.M_{12}$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/4$	4	$\cong T(M_{12})$	2.Ru	13	$\mathbb{Z}/12$	12	$\cong T(\mathrm{Ru})$
M_{12}	11	$\mathbb{Z}/5$	5	$\mathbb{Z}/10$	Ru	29	$\mathbb{Z}/14$	14	$\mathbb{Z}/28$
$2.M_{12}$	11	$\mathbb{Z}/10$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/10$	2.Ru	29	$\mathbb{Z}/28$	14	$\mathbb{Z}/2\oplus\mathbb{Z}/28$
J_1	3	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	2	$\mathbb{Z}/2\oplus\mathbb{Z}/4$	Suz	7	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$
J_1	5	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	2	$\mathbb{Z}/2\oplus\mathbb{Z}/4$	2.Suz	7	$\mathbb{Z}/6$	6	$\cong T(\mathrm{Suz})$
J_1	7	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$	3.Suz	7	$\mathbb{Z}/3\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/3\oplus\mathbb{Z}/12$
J_1	11	$\mathbb{Z}/10$	10	$\mathbb{Z}/20$	6.Suz	7	$\mathbb{Z}/3\oplus\mathbb{Z}/6$	6	$\cong T(3.\mathrm{Suz})$
J_1	19	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$	Suz	11	$\mathbb{Z}/10$	10	$\mathbb{Z}/20$
M_{22}	5	$\mathbb{Z}/4$	4	$\mathbb{Z}/8$	2.Suz	11	$\mathbb{Z}/20$	10	$\mathbb{Z}/2\oplus\mathbb{Z}/20$
$2.M_{22}$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/4$	4	$\mathbb{Z}/2\oplus\mathbb{Z}/8$	3.Suz	11	$\mathbb{Z}/30$	10	$\mathbb{Z}/60$
$3.M_{22}$	5	$\mathbb{Z}/12$	4	$\mathbb{Z}/24$	6.Suz	11	$\mathbb{Z}/60$	10	$\mathbb{Z}/2\oplus\mathbb{Z}/60$
$4.M_{22}$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/8$	4	$\mathbb{Z}/4\oplus\mathbb{Z}/8$	Suz	13	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$
$6.M_{22}$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/12$	4	$\mathbb{Z}/2\oplus\mathbb{Z}/24$	2.Suz	13	$\mathbb{Z}/12$	6	$\mathbb{Z}/2\oplus\mathbb{Z}/12$
$12.M_{22}$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/24$	4	$\mathbb{Z}/4\oplus\mathbb{Z}/24$	3.Suz	13	$\mathbb{Z}/3\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/3\oplus\mathbb{Z}/12$
M_{22}	7	$\mathbb{Z}/3$	3	$\mathbb{Z}/6$	6.Suz	13	$\mathbb{Z}/3\oplus\mathbb{Z}/12$	6	$\mathbb{Z}/6\oplus\mathbb{Z}/12$
$2.M_{22}$	7	$\mathbb{Z}/6$	3	$\mathbb{Z}/2\oplus\mathbb{Z}/6$	ON	5	$\mathbb{Z}/2\oplus\mathbb{Z}/4$	4	$\mathbb{Z}/2\oplus\mathbb{Z}/8$
$3.M_{22}$	7	$\mathbb{Z}/3\oplus\mathbb{Z}/3$	3	$\mathbb{Z}/3\oplus\mathbb{Z}/6$	3.ON	5	$\mathbb{Z}/2\oplus\mathbb{Z}/4$	4	$\cong T(\mathrm{ON})$
$4.M_{22}$	7	$\mathbb{Z}/12$	3	$\mathbb{Z}/2\oplus\mathbb{Z}/12$	ON	11	$\mathbb{Z}/10$	10	$\mathbb{Z}/20$
$6.M_{22}$	7	$\mathbb{Z}/3\oplus\mathbb{Z}/6$	3	$\mathbb{Z}/6\oplus\mathbb{Z}/6$	3.ON	11	$\mathbb{Z}/30$	10	$\mathbb{Z}/60$
$12.M_{22}$	7	$\mathbb{Z}/3\oplus\mathbb{Z}/12$	3	$\mathbb{Z}/6\oplus\mathbb{Z}/12$	ON	19	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$
M_{22}	11	$\mathbb{Z}/5$	5	$\mathbb{Z}/10$	3.ON	19	$\mathbb{Z}/3\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/3\oplus\mathbb{Z}/12$
$2.M_{22}$	11	$\mathbb{Z}/10$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/10$	ON	31	$\mathbb{Z}/15$	15	$\mathbb{Z}/30$
$3.M_{22}$	11	$\mathbb{Z}/15$	5	$\mathbb{Z}/30$	3.ON	31	$\mathbb{Z}/3\oplus\mathbb{Z}/15$	15	$\mathbb{Z}/3\oplus\mathbb{Z}/30$
$4.M_{22}$	11	$\mathbb{Z}/20$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/20$	Co_3	7	$\mathbb{Z}/2\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/2\oplus\mathbb{Z}/12$
$6.M_{22}$	11	$\mathbb{Z}/30$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/30$	Co_3	11	$\mathbb{Z}/10$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/10$
$12.M_{22}$	11	$\mathbb{Z}/60$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/60$	Co_3	23	$\mathbb{Z}/11$	11	$\mathbb{Z}/22$
J_2	7	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$	Co_2	7	$\mathbb{Z}/2\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/2\oplus\mathbb{Z}/12$
$2.J_2$	7	$\mathbb{Z}/12$	6	$\mathbb{Z}/2\oplus\mathbb{Z}/12$	Co_2	11	$\mathbb{Z}/10$	10	$\mathbb{Z}/20$
M_{23}	5	$\mathbb{Z}/4$	4	$\mathbb{Z}/8$	Co_2	23	$\mathbb{Z}/11$	11	$\mathbb{Z}/22$
M_{23}	7	$\mathbb{Z}/6$	3	$\mathbb{Z}/2\oplus\mathbb{Z}/6$	Fi_{22}	7	$\mathbb{Z}/2\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/2\oplus\mathbb{Z}/12$
M_{23}	11	$\mathbb{Z}/5$	5	$\mathbb{Z}/10$	$2.Fi_{22}$	7	$\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/12$
M_{23}	23	$\mathbb{Z}/11$	11	$\mathbb{Z}/22$	$3.Fi_{22}$	7	$\mathbb{Z}/6\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/6\oplus\mathbb{Z}/12$
HS	7	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$	$6.Fi_{22}$	7	$\mathbb{Z}/2\oplus\mathbb{Z}/6\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/2\oplus\mathbb{Z}/6\oplus\mathbb{Z}/12$
2.HS	7	$\mathbb{Z}/12$	6	$\mathbb{Z}/2\oplus\mathbb{Z}/12$	Fi_{22}	11	$\mathbb{Z}/10$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/10$
HS	11	$\mathbb{Z}/5$	5	$\mathbb{Z}/10$	$2.Fi_{22}$	11	$\mathbb{Z}/2\oplus\mathbb{Z}/10$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/10$
2.HS	11	$\mathbb{Z}/10$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/10$	$3.Fi_{22}$	11	$\mathbb{Z}/30$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/30$
J_3	5	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	2	$\mathbb{Z}/2\oplus\mathbb{Z}/4$	$6.Fi_{22}$	11	$\mathbb{Z}/2\oplus\mathbb{Z}/30$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/30$
$3.J_3$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/6$	2	$\mathbb{Z}/2\oplus\mathbb{Z}/12$	Fi_{22}	13	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$
J_3	17	$\mathbb{Z}/8$	8	$\mathbb{Z}/16$	$2.Fi_{22}$	13	$\mathbb{Z}/2\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/2\oplus\mathbb{Z}/12$
$3.J_3$	17	$\mathbb{Z}/24$	8	$\mathbb{Z}/48$	$3.Fi_{22}$	13	$\mathbb{Z}/3\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/3\oplus\mathbb{Z}/12$
J_3	19	$\mathbb{Z}/9$	9	$\mathbb{Z}/18$	$6.Fi_{22}$	13	$\mathbb{Z}/6\oplus\mathbb{Z}/6$	6	$\mathbb{Z}/6\oplus\mathbb{Z}/12$
$3.J_3$	19	$\mathbb{Z}/3\oplus\mathbb{Z}/9$	9	$\mathbb{Z}/3\oplus\mathbb{Z}/18$	HN	7	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$
M_{24}	5	$\mathbb{Z}/4$	4	$\mathbb{Z}/8$	HN	11	$\mathbb{Z}/2\oplus\mathbb{Z}/10$	10	$\mathbb{Z}/2\oplus\mathbb{Z}/20$
M_{24}	7	$\mathbb{Z}/6$	3	$\mathbb{Z}/2\oplus\mathbb{Z}/6$	HN	19	$\mathbb{Z}/9$	9	$\mathbb{Z}/18$
M_{24}	11	$\mathbb{Z}/10$	10	$\mathbb{Z}/20$	Ly	7	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$
M_{24}	23	$\mathbb{Z}/11$	11	$\mathbb{Z}/22$	Ly	11	$\mathbb{Z}/10$	5	$\mathbb{Z}/2\oplus\mathbb{Z}/10$
McL	7	$\mathbb{Z}/6$	3	$\mathbb{Z}/2\oplus\mathbb{Z}/6$	Ly	31	$\mathbb{Z}/6$	6	$\mathbb{Z}/12$
3.McL	7	$\mathbb{Z}/3\oplus\mathbb{Z}/6$	3	$\mathbb{Z}/6\oplus\mathbb{Z}/6$	Ly	37	$\mathbb{Z}/18$	18	$\mathbb{Z}/36$
McL	11	$\mathbb{Z}/5$	5	$\mathbb{Z}/10$	Ly	67	$\mathbb{Z}/22$	22	$\mathbb{Z}/44$
3.McL	11	$\mathbb{Z}/15$	5	$\mathbb{Z}/30$	Th	13	$\mathbb{Z}/12$	12	$\mathbb{Z}/24$

G	p	$X(H)$	e	$T(G)$	G	p	$X(H)$	e	$T(G)$
Th	19	$\mathbb{Z}/18$	18	$\mathbb{Z}/36$	Fi_{24}'	23	$\mathbb{Z}/11$	11	$\mathbb{Z}/22$
Th	31	$\mathbb{Z}/15$	15	$\mathbb{Z}/30$	$3.\mathrm{Fi}_{24}'$	23	$\mathbb{Z}/33$	11	$\mathbb{Z}/66$
Fi_{23}	7	$\mathbb{Z}/2 \oplus \mathbb{Z}/6$	6	$\mathbb{Z}/2 \oplus \mathbb{Z}/12$	Fi_{24}'	29	$\mathbb{Z}/14$	14	$\mathbb{Z}/28$
Fi_{23}	11	$\mathbb{Z}/2 \oplus \mathbb{Z}/10$	10	$\mathbb{Z}/2 \oplus \mathbb{Z}/20$	$3.\mathrm{Fi}_{24}'$	29	$\mathbb{Z}/42$	14	$\mathbb{Z}/84$
Fi_{23}	13	$\mathbb{Z}/2 \oplus \mathbb{Z}/6$	6	$\mathbb{Z}/2 \oplus \mathbb{Z}/12$	B	11	$\mathbb{Z}/2 \oplus \mathbb{Z}/10$	10	$\mathbb{Z}/2 \oplus \mathbb{Z}/20$
Fi_{23}	17	$\mathbb{Z}/16$	16	$\mathbb{Z}/32$	2.B	11	$\mathbb{Z}/2 \oplus \mathbb{Z}/10$	10	$\cong T(\mathrm{B})$
Fi_{23}	23	$\mathbb{Z}/11$	11	$\mathbb{Z}/22$	B	13	$\mathbb{Z}/2 \oplus \mathbb{Z}/12$	12	$\mathbb{Z}/2 \oplus \mathbb{Z}/24$
Co_1	11	$\mathbb{Z}/2 \oplus \mathbb{Z}/10$	10	$\mathbb{Z}/2 \oplus \mathbb{Z}/20$	2.B	13	$\mathbb{Z}/2 \oplus \mathbb{Z}/12$	12	$\cong T(\mathrm{B})$
$2.\mathrm{Co}_1$	11	$\mathbb{Z}/2 \oplus \mathbb{Z}/10$	10	$\cong T(\mathrm{Co}_1)$	B	17	$\mathbb{Z}/2 \oplus \mathbb{Z}/16$	16	$\mathbb{Z}/2 \oplus \mathbb{Z}/32$
Co_1	13	$\mathbb{Z}/12$	12	$\mathbb{Z}/24$	2.B	17	$\mathbb{Z}/2 \oplus \mathbb{Z}/16$	16	$\cong T(\mathrm{B})$
$2.\mathrm{Co}_1$	13	$\mathbb{Z}/12$	12	$\cong T(\mathrm{Co}_1)$	B	19	$\mathbb{Z}/2 \oplus \mathbb{Z}/18$	18	$\mathbb{Z}/2 \oplus \mathbb{Z}/36$
Co_1	23	$\mathbb{Z}/11$	11	$\mathbb{Z}/22$	2.B	19	$\mathbb{Z}/2 \oplus \mathbb{Z}/18$	18	$\cong T(\mathrm{B})$
$2.\mathrm{Co}_1$	23	$\mathbb{Z}/22$	11	$\mathbb{Z}/2 \oplus \mathbb{Z}/22$	B	23	$\mathbb{Z}/22$	11	$\mathbb{Z}/2 \oplus \mathbb{Z}/22$
J_4	5	$\mathbb{Z}/4$	4	$\mathbb{Z}/8$	2.B	23	$\mathbb{Z}/2 \oplus \mathbb{Z}/22$	11	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/22$
J_4	7	$\mathbb{Z}/6$	3	$\mathbb{Z}/2 \oplus \mathbb{Z}/6$	B	31	$\mathbb{Z}/15$	15	$\mathbb{Z}/30$
J_4	23	$\mathbb{Z}/22$	22	$\mathbb{Z}/44$	2.B	31	$\mathbb{Z}/30$	15	$\mathbb{Z}/2 \oplus \mathbb{Z}/30$
J_4	29	$\mathbb{Z}/28$	28	$\mathbb{Z}/56$	B	47	$\mathbb{Z}/23$	23	$\mathbb{Z}/46$
J_4	31	$\mathbb{Z}/10$	10	$\mathbb{Z}/20$	2.B	47	$\mathbb{Z}/46$	23	$\mathbb{Z}/2 \oplus \mathbb{Z}/46$
J_4	37	$\mathbb{Z}/12$	12	$\mathbb{Z}/24$	M	17	$\mathbb{Z}/16$	16	$\mathbb{Z}/32$
J_4	43	$\mathbb{Z}/14$	14	$\mathbb{Z}/28$	M	19	$\mathbb{Z}/18$	18	$\mathbb{Z}/36$
Fi_{24}'	11	$\mathbb{Z}/10$	10	$\mathbb{Z}/20$	M	23	$\mathbb{Z}/22$	11	$\mathbb{Z}/2 \oplus \mathbb{Z}/22$
$3.\mathrm{Fi}_{24}'$	11	$\mathbb{Z}/30$	10	$\mathbb{Z}/60$	M	29	$\mathbb{Z}/28$	28	$\mathbb{Z}/56$
Fi_{24}'	13	$\mathbb{Z}/2 \oplus \mathbb{Z}/12$	12	$\mathbb{Z}/2 \oplus \mathbb{Z}/24$	M	31	$\mathbb{Z}/30$	15	$\mathbb{Z}/2 \oplus \mathbb{Z}/30$
$3.\mathrm{Fi}_{24}'$	13	$\mathbb{Z}/2 \oplus \mathbb{Z}/12$	12	$\cong T(\mathrm{Fi}_{24}')$	M	41	$\mathbb{Z}/40$	40	$\mathbb{Z}/80$
Fi_{24}'	17	$\mathbb{Z}/16$	16	$\mathbb{Z}/32$	M	47	$\mathbb{Z}/46$	23	$\mathbb{Z}/2 \oplus \mathbb{Z}/46$
$3.\mathrm{Fi}_{24}'$	17	$\mathbb{Z}/48$	16	$\mathbb{Z}/96$	M	59	$\mathbb{Z}/29$	29	$\mathbb{Z}/58$
					M	71	$\mathbb{Z}/35$	35	$\mathbb{Z}/70$

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