UNIVERSAL p'-CENTRAL EXTENSIONS

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Abstract. It is well-known that a finite group possesses a universal central extension if and only if it is a perfect group. Similarly, given a prime number p, we show that a finite group possesses a universal p'-central extension if and only if the p'-part of its abelianization is trivial. This question arises naturally when working with group representations over a field of characteristic p.

1. Introduction

The theory of central extensions of groups is well-known and goes back to the work of Schur. It is tightly connected with the theory of projective representations of groups over the field $\mathbb{C}$ of complex numbers. The link is provided by the most typical central extension of groups, namely

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{PGL}_n(\mathbb{C}) \longrightarrow 1\,,$$

where n is a positive integer and $\mathbb{C}^\times$ denotes the group of nonzero elements in $\mathbb{C}$.

When working with representations of groups over a field k of characteristic p, a similar sequence occurs with k instead of $\mathbb{C}$. The main difference is that the group $k^\times$ does not contain any nontrivial element of order a power of p. In other words, the torsion subgroup of $k^\times$ (i.e. the group of roots of unity in k) is a p'-group, in the sense that each of its elements has order prime to p.

One of our purposes here is to extend Schur's theory by proving similar results when k is an algebraically closed field of characteristic p. We shall restrict to the case of finite groups, as Schur did (thus leaving aside the important theory of the Steinberg group and its use in algebraic K-theory, see [5, §5]). Given a finite group G, Schur proved that G has a universal central extension if and only if G is a perfect group. For our purpose, we need to replace central extensions by p'-central extensions. By a p'-central extension of G, we mean a group extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1\,,$$

where $Z = \operatorname{Ker}\pi$ is a finite group of order prime to p, contained in the centre of E. We also need to replace perfect groups by p'-perfect groups. By a p'-perfect group, we mean a finite group G whose abelianization $G/[G,G]$ has a trivial p'-part, i.e. is a p-group.

One of our main results asserts that G has a universal p'-central extension if and only if G is a p'-perfect group. As in the case of Schur's theory, we have to deal with the problem of lifting projective representations of G to ordinary representations of a group $\widetilde{G}$, a

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so-called representation group of G. But we need to replace representation groups by p'-representation groups (defined in Section 4), because we work with representations of groups over a field k of characteristic p instead of $\mathbb{C}$.

We were lead to the questions discussed in this paper by some work we did on endo-trivial representations of finite groups in characteristic p. In [4, Section 6], we mention a special case of the above result, namely that any prefect group has a universal p'-central extension. This provides a motivation for the present article.

Another motivation is simply to fill a gap by slightly extending a classical theory. Many of our results either are direct analogues of well-known facts about central extensions or are inspired by well-known situations about representations in characteristic zero. An exception is Proposition 5.2, which requires some more work.

We note that in the preparation of this text we used [1, §11], [5, §5], [6, §3.5], [7, Chapters 7 & 11] and [8, §6.9] for classical results on central extensions.

Finally, we point out that the standard textbook [1] uses different notions of *universality* for central extensions. For completeness the last section of this note is devoted to describing how these different notions of universality as well as the projective lifting property are linked to each other in the case of p'-perfect groups.

Throughout this paper, G denotes a finite group and k denotes an algebraically closed field of prime characteristic p. Moreover, $k^\times$ denotes the multiplicative group of nonzero elements of $k^\times$. A finite group is called a *p'-group* if its order is prime to p. If A is a finite abelian group, we denote by $A_{p'}$ its p'-part (namely $A \cong A_p \times A_{p'}$, where A_p is a p-group and $A_{p'}$ is a p'-group).

2. Universal p'-central extensions and p'-perfect groups

By a *central extension* (E, π) of G, we mean a group extension of the form

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1\,,$$

where $Z = \operatorname{Ker} \pi$ and Z is contained in the centre of E. A central extension (E, π) of G is called a *p'-central extension* provided $\operatorname{Ker} \pi$ is a finite p'-group. In that case, E is finite because both Z and G are.

The two central concepts of this note are the following:

Definition 2.1. A *universal p'-central extension* of a finite group G is a p'-central extension

$$1 \longrightarrow M \longrightarrow E \xrightarrow{\nu} G \longrightarrow 1$$

with the following universal property: for any p'-central extension

$$1 \longrightarrow Z \longrightarrow X \xrightarrow{\pi} G \longrightarrow 1\,,$$

there exists a unique group homomorphism $\phi : E \longrightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{\nu} & G & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \phi|_M} & & \Big\downarrow{\scriptstyle \exists!\phi} & & \Big\| & & \\
1 & \longrightarrow & Z & \longrightarrow & X & \xrightarrow{\pi} & G & \longrightarrow & 1
\end{array}$$

Definition 2.2. A finite group G is called *p'-perfect* if the p'-part of the abelianization of G is trivial, i.e. $(G/[G,G])_{p'} = \{1\}$. In other words, the abelianization $G/[G,G]$ is a p-group.

For ease of notation we define the finite abelian group

$$X(G) := \mathrm{Hom}(G, k^\times)\,.$$

Since k is algebraically closed, the torsion subgroup of $k^\times$ is the group μ_k of all p'-roots of unity. Therefore any group homomorphism $G \longrightarrow k^\times$ factors through $G \longrightarrow (G/[G,G])_{p'}$ and also through the inclusion $\mu_k \longrightarrow k^\times$. Thus we see that

$$X(G) \cong \mathrm{Hom}(G/[G,G])_{p'}, \mu_k)$$

and this is isomorphic to the dual group of $(G/[G,G])_{p'}$. Since any finite abelian group is isomorphic to its dual (but not canonically), we actually have an isomorphism $X(G) \cong (G/[G,G])_{p'}$. Therefore, we obtain the following characterization:

Lemma 2.3. *G is p'-perfect if and only if $X(G) = \{1\}$.*

Lemma 2.4. *If $1 \longrightarrow M \longrightarrow E \xrightarrow{\nu} G \longrightarrow 1$ is a universal p'-central extension of G, then both E and G are p'-perfect groups.*

Proof. The homomorphism ν induces a homomorphism $\mathrm{Inf}_G^E : X(G) \longrightarrow X(E)$ (called inflation, see Section 3) which is clearly injective. Therefore, it suffices to prove that $X(E) = \{1\}$, that is, $(E/[E,E])_{p'} = \{1\}$. Let

$$\eta : E \longrightarrow (E/[E,E])_{p'}$$

denote the quotient map and let 1 denote the trivial map $E \longrightarrow (E/[E,E])_{p'}$. We obtain two homomorphisms (η, ν) and $(1, \nu)$ making the following diagram commute, respectively:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{\nu} & G & \longrightarrow & 1 \\
 & & {\scriptstyle\eta}\downarrow\downarrow{\scriptstyle 1} & & {\scriptstyle(\eta,\nu)}\downarrow\downarrow{\scriptstyle(1,\nu)} & & \| & & \\
1 & \longrightarrow & (E/[E,E])_{p'} & \longrightarrow & (E/[E,E])_{p'} \times G & \xrightarrow{pr_G} & G & \longrightarrow & 1\,,
\end{array}$$

where pr_G denotes the canonical projection onto G. By the uniqueness condition in the universal property of (E, ν), we conclude that $\eta = 1$. Therefore $(E/[E,E])_{p'} = \{1\}$. □

Lemma 2.5. *Let $1 \longrightarrow M \longrightarrow E \xrightarrow{\nu} G \longrightarrow 1$ be a p'-central extension of G, let $1 \longrightarrow Z \longrightarrow X \xrightarrow{\pi} G^* \longrightarrow 1$ be any central extension, and let $\theta : G \longrightarrow G^*$ be a group homomorphism. Suppose that E is p'-perfect and that Z has no p-torsion (i.e. Z has no nontrivial element of order a power of p). Then there exists at most one group homomorphism $\phi : E \longrightarrow X$ making the following diagram commute:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{\nu} & G & \longrightarrow & 1 \\
 & & {\scriptstyle\phi|_M}\downarrow & & {\scriptstyle\phi}\downarrow & & {\scriptstyle\theta}\downarrow & & \\
1 & \longrightarrow & Z & \longrightarrow & X & \xrightarrow{\pi} & G^* & \longrightarrow & 1
\end{array}$$

Proof. Suppose that there exist two group homomorphisms $\phi, \psi : E \longrightarrow X$ making the following diagram commute, respectively:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{\nu} & G & \longrightarrow & 1 \\ & & {\scriptstyle \phi|_M}\downarrow\ \downarrow{\scriptstyle \psi|_M} & & {\scriptstyle \phi}\downarrow\ \downarrow{\scriptstyle \psi} & & {\scriptstyle \theta}\downarrow & & \\ 1 & \longrightarrow & Z & \longrightarrow & X & \xrightarrow{\pi} & G^* & \longrightarrow & 1 \end{array}$$

We define a map $\lambda : E \longrightarrow Z$ by $\lambda(e) := \phi(e)\psi(e)^{-1}$ for every $e \in E$. Since Z is central, for every $e_1, e_2 \in E$ we have:

$$\begin{aligned} \phi(e_1e_2) = \phi(e_1)\phi(e_2) &= \lambda(e_1)\psi(e_1)\lambda(e_2)\psi(e_2) \\ &= \lambda(e_1)\lambda(e_2)\psi(e_1)\psi(e_2) = \lambda(e_1)\lambda(e_2)\psi(e_1e_2) \end{aligned}$$

Hence $\lambda(e_1)\lambda(e_2) = \lambda(e_1e_2)$ and so λ is a homomorphism. As Z is an abelian group which has no p-torsion, the homomorphism λ must factor through the quotient $(E/[E,E])_{p'}$. But E is p'-perfect, that is, $(E/[E,E])_{p'} = \{1\}$. Therefore $\lambda(e) = 1$ for all $e \in E$, so that $\phi = \psi$. □

Proposition 2.6. *Let* $1 \longrightarrow M \longrightarrow E \xrightarrow{\nu} G \longrightarrow 1$ *be a* p'*-central extension of* G*. Assume moreover that* E *fulfills the following two conditions:*

1. E *is* p'*-perfect.*
2. *Every* p'*-central extension of* E *splits.*

Then (E, ν) *is a universal* p'*-central extension of* G*.*

Proof. Given a p'-central extension $1 \longrightarrow Z \longrightarrow X \xrightarrow{\pi} G \longrightarrow 1$, construct a homomorphism $E \longrightarrow X$ as follows. Take the pullback

$$E \times_G X = \{(e, x) \in E \times X \mid \nu(e) = \pi(x)\}$$

of ν and π. Then

$$1 \longrightarrow Z \longrightarrow E \times_G X \xrightarrow{pr_E} E \longrightarrow 1$$

is a p'-central extension of E. Therefore it splits by assumption, so there exists a group homomorphism $\sigma : E \longrightarrow E \times_G X$ such that $pr_E \circ \sigma = \mathrm{Id}_E$.

Now $\phi := pr_X \circ \sigma : E \longrightarrow X$ makes the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{\nu} & G & \longrightarrow & 1 \\ & & {\scriptstyle \phi|_M}\downarrow & & {\scriptstyle \phi}\downarrow & & \| & & \\ 1 & \longrightarrow & Z & \longrightarrow & X & \xrightarrow{\pi} & G & \longrightarrow & 1 \end{array}$$

commute, because $\pi \circ \phi = \pi \circ pr_X \circ \sigma = \nu \circ pr_E \circ \sigma = \nu \circ \mathrm{Id}_E = \nu$. Since E is p'-perfect and Z is a p'-group, ϕ is unique, by Lemma 2.5. This proves that (E, ν) is a universal p'-central extension of G. □

We shall prove later (Theorem 7.1) that the converse of Proposition 2.6 holds as well.

3. Central extensions and the 5-term exact sequence

We assume the reader is familiar with standard concepts of the theory of cohomology of groups and its connections to group extensions. We refer to [1, §8], [7, Chapter 7], or [8, Chapter 6] for background material on this topic. We denote, respectively, by $H^1(G,B)$ and $H^2(G,B)$, the first and second cohomology groups of G, with coefficients in an abelian group B viewed as a G-module with trivial G-action. Since B has trivial action, $H^1(G,B) = \mathrm{Hom}(G,B)$. We shall later use mainly the finite group $H^2(G,k^\times)$.

It is a standard result of the theory of cohomology of groups that to each central extension (E,π) of G with $Z := \mathrm{Ker}(\pi)$ and each abelian multiplicative group B can be associated a 5-term exact sequence

$$1 \longrightarrow \mathrm{Hom}(G,B) \xrightarrow{\mathrm{Inf}_G^E} \mathrm{Hom}(E,B) \xrightarrow{\mathrm{Res}_Z^E} \mathrm{Hom}(Z,B) \xrightarrow{\mathrm{tr}} H^2(G,B) \xrightarrow{\mathrm{Inf}_G^E} H^2(E,B)$$

This is also called the *Hochschild-Serre exact sequence* in the literature, because it arises from the low degree terms in the Hochschild-Serre spectral sequence associated to the given central extension (see [8, Section 6.8]). The first homomorphism Inf_G^E is the inflation of homomorphisms, which is defined by

$$\mathrm{Inf}_G^E(\psi) = \psi \circ \pi \ \text{ for each } \psi \in \mathrm{Hom}(G,B)\,,$$

and is clearly injective. The homomorphism Res_Z^E denotes the ordinary restriction of maps from E to Z. The last homomorphism Inf_G^E is the inflation in cohomology, which we now recall: given a class $[\alpha] \in H^2(G,B)$ represented by a 2-cocycle $\alpha \in Z^2(G,B)$, the element $\mathrm{Inf}_G^E([\alpha]) \in H^2(E,B)$ is the cohomology class represented by the 2-cocycle $\beta \in Z^2(E,B)$ defined by

$$\beta(u,v) := \alpha(\pi(u),\pi(v)),\ \forall\, u,v \in E\,.$$

Finally, the homomorphism tr is called *transgression* and is defined as follows: given $\varphi \in \mathrm{Hom}(Z,B)$, then

$$\mathrm{tr}(\varphi) := [\varphi \circ \alpha] \in H^2(G,B)\,,$$

where $\alpha \in Z^2(G,Z)$ is a 2-cocycle in the cohomology class corresponding to the central extension (E,π). We also refer the reader to [1, §11E] for further details on the transgression map.

For ease of notation we simply write Inf for the inflation maps and Res for the restriction map. We shall apply this sequence only to the case $B = k^\times$. Since $\mathrm{Hom}(G,k^\times) = X(G)$ in our notation, the 5-term exact sequence takes the form:

$$1 \longrightarrow X(G) \xrightarrow{\mathrm{Inf}} X(E) \xrightarrow{\mathrm{Res}} X(Z) \xrightarrow{\mathrm{tr}} H^2(G,k^\times) \xrightarrow{\mathrm{Inf}} H^2(E,k^\times)$$

We see at once from the injectivity of the first inflation map that if E is p'-perfect, then so is G.

4. The notion of a p'-representation group

We recall that a central extension (E,π) of G is said to have the *projective lifting property (relative to k)* if, for every finite-dimensional k-vector space V, every group homomorphism $\theta : G \longrightarrow \mathrm{PGL}(V)$ can be completed to a commutative diagram of group homomorphisms:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & E & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \widetilde{\theta}|_Z} & & \downarrow{\scriptstyle \widetilde{\theta}} & & \downarrow{\scriptstyle \theta} & & \\ 1 & \longrightarrow & k^\times \cdot \mathrm{Id}_V & \longrightarrow & \mathrm{GL}(V) & \xrightarrow{\pi_V} & \mathrm{PGL}(V) & \longrightarrow & 1 \end{array}$$

In general, the homomorphism $\widetilde{\theta}$ is not uniquely defined. However, by Lemma 2.5, $\widetilde{\theta}$ is unique if E is p'-perfect, because $k^\times$ has no p-torsion.

Definition 4.1. A *p'-representation group of G* (or a *representation group of G relative to k*) is a p'-central extension $(\widetilde{G}, \pi_{\widetilde{G}})$ of G of minimal order with the projective lifting property.

We set $M_k(G) := H^2(G, k^\times)$ and recall that this group is isomorphic the p'-part of $H^2(G, \mathbb{C}^\times)$, the ordinary Schur multiplier of G (see e.g. [3, Proposition 2.2.14]). A p'-representation group of G can be characterized as follows.

Theorem 4.2 ([6, Theorem 3.5.2])**.** *Let $(\widetilde{G}, \pi_{\widetilde{G}})$ be a p'-central extension of G with kernel $Z := \mathrm{Ker}(\pi_{\widetilde{G}})$. Then $(\widetilde{G}, \pi_{\widetilde{G}})$ is a p'-representation group of G if and only if the following two conditions hold:*

(a) *The transgression* $\mathrm{tr} : X(Z) \longrightarrow H^2(G, k^\times)$ *is injective, or, equivalently,* $Z \leq [\widetilde{G}, \widetilde{G}]$.

(b) $|Z| = |M_k(G)|$.

If this is the case, then we have $Z \cong M_k(G)$ and the transgression map tr *is an isomorphism.*

Schur proved that a representation group of a finite group G (that is, a representation group of G relative to $\mathbb{C}$) always exists. His construction uses a free presentation of the group G (and gives rise to Hopf's formula for the Schur multiplier). An alternative construction, also valid over fields of prime characteristic, was given by Yamazaki [9]. This alternative approach can also be found in [1, §11] and [6, §5.5].

Theorem 4.3. *Given a finite group G, a p'-representation group of G exists.*

By Theorem 4.2, such a p'-representation group of G has the form

$$1 \longrightarrow M_k(G) \longrightarrow \widetilde{G} \xrightarrow{\pi_{\widetilde{G}}} G \longrightarrow 1$$

and the associated transgression map

$$\mathrm{tr} : X(M_k(G)) \longrightarrow M_k(G) = H^2(G, k^\times)$$

is an isomorphism. We also emphasize the following:

Lemma 4.4. *Let $(\widetilde{G}, \pi_{\widetilde{G}})$ be a p'-representation group of G. Then*

$$X(\widetilde{G}) = \mathrm{Inf}_G^{\widetilde{G}}(X(G)) \cong X(G)\,.$$

Proof. Let $Z =: \mathrm{Ker}(\pi_{\widetilde{G}})$. Since $Z \leq [\widetilde{G}, \widetilde{G}]$ by Theorem 4.2, any homomorphism $\chi : \widetilde{G} \to k^\times$ has Z in its kernel. Thus χ is in the image of the inflation map $X(G) \xrightarrow{\mathrm{Inf}} X(\widetilde{G})$. □

It is well known that the representation groups of G are in general not unique, although it is the case if G is perfect. Similarly, the p'-representation groups of G are in general not unique, although it is the case if G is p'-perfect (see Corollary 6.3).

5. Cohomological criteria

We first show that, in order to check condition (2) of Proposition 2.6, it is equivalent to check that $H^2(E, k^\times) = \{1\}$.

Proposition 5.1. *Let E be a p'-perfect finite group. Then the following conditions are equivalent:*

(a) $H^2(E, k^\times) = \{1\}$.
(b) $H^2(E, A) = \{1\}$ *for any finite abelian group A of order prime to p with trivial action of E.*
(c) *Any p'-central extension $1 \longrightarrow A \longrightarrow F \longrightarrow E \longrightarrow 1$ of E splits.*

Proof.
(b) $\Leftrightarrow$ (c): This is a classical result of the theory of cohomology of groups. See e.g. [7, Theorem 7.34].
(c) $\Rightarrow$ (a): By (c), a p'-representation group of E splits, hence has the form

$$1 \longrightarrow M \longrightarrow M \times E \xrightarrow{pr_E} E \longrightarrow 1$$

where $M = M_k(E) = H^2(E, k^\times)$. This is a p'-central extension of E of minimal order with the projective lifting property. But the extension (E, Id_E) also has the projective lifting property, because E embeds in $M \times E$ via $e \mapsto (1, e)$. By minimality we have $\{1\} = M = H^2(E, k^\times)$.
(a) $\Rightarrow$(b): First we claim that if $H^2(E, k^\times) = \{1\}$, then $H^2(E, \prod_{i=1}^n k^\times) = \{1\}$ for any positive integer n. This is easily seen using an induction argument on n. Indeed, if the claim holds for $n-1$, then the short exact sequence of coefficients

$$1 \longrightarrow k^\times \longrightarrow \prod_{i=1}^{n} k^\times \longrightarrow \prod_{i=1}^{n-1} k^\times \longrightarrow 1$$

yields a long exact sequence in cohomology:

$$\cdots \longrightarrow H^2(E, k^\times) = \{1\} \longrightarrow H^2(E, \prod_{i=1}^{n} k^\times) \longrightarrow H^2(E, \prod_{i=1}^{n-1} k^\times) = \{1\} \longrightarrow \cdots$$

The claim follows.
Next we prove that if A is a finite abelian group of order prime to p, then $H^2(E, A) = \{1\}$. Indeed, A is isomorphic to a finite product of cyclic groups of order prime to p, hence can be embedded in $\prod_{i=1}^n k^\times$ for some $n \geq 1$. This yields a long exact sequence in cohomology:

$$\cdots \longrightarrow H^1\big(E, (\prod_{i=1}^{n} k^\times)/A\big) \longrightarrow H^2(E, A) \longrightarrow H^2(E, \prod_{i=1}^{n} k^\times) \longrightarrow \cdots$$

But $H^1\big(E, (\prod_{i=1}^n k^\times)/A\big) = \{1\}$ since E is p'-perfect and $H^2(E, \prod_{i=1}^n k^\times) = \{1\}$ by the first claim. Therefore $H^2(E, A) = \{1\}$. □

Now we come to the crucial result, which is a consequence of a result of Iwahori-Matsumoto [2, Prop.1.3] stating that the 5-term exact sequence can be extended to a 6-term exact sequence.

Proposition 5.2. *Let G be a p'-perfect finite group and $(\widetilde{G}, \pi_{\widetilde{G}})$ be a p'-representation group of G. Then $H^2(\widetilde{G}, k^\times) = \{1\}$.*

Proof. The 5-term exact sequence associated to $(\widetilde{G}, \pi_{\widetilde{G}})$ has the form:

$$1 \longrightarrow X(G) \longrightarrow X(\widetilde{G}) \longrightarrow X(M_k(G)) \underset{\cong}{\overset{\mathrm{tr}}{\longrightarrow}} H^2(G, k^\times) \overset{\mathrm{Inf}}{\longrightarrow} H^2(\widetilde{G}, k^\times)$$

where $X(\widetilde{G}) \cong X(G)$ by Lemma 4.4 (and they are actually both trivial, by the assumption that G is p'-perfect). Moreover, the transgression map

$$\mathrm{tr} : X(M_k(G)) \longrightarrow H^2(G, k^\times) = M_k(G)$$

is an isomorphism, by Theorem 4.2. Therefore the inflation

$$\mathrm{Inf}_G^{\widetilde{G}} : H^2(G, k^\times) \longrightarrow H^2(\widetilde{G}, k^\times)$$

is the trivial map.

Now, since $k^\times$ is a divisible abelian group, it follows from [2, Prop.1.3] that the 5-term exact sequence can be extended to a 6-term exact sequence in the following way:

$$\cdots \overset{\mathrm{tr}}{\longrightarrow} H^2(G, k^\times) \overset{\mathrm{Inf}}{\longrightarrow} H^2(\widetilde{G}, k^\times) \overset{\theta_2}{\longrightarrow} P(\widetilde{G}, M_k(G); k^\times)$$

where $P\big(\widetilde{G}, M_k(G); k^\times\big)$ is the group of pairings $f : \widetilde{G} \times M_k(G) \longrightarrow k^\times$ (see [2, §1]) and θ_2 is defined by

$$\theta_2([\alpha])(x, z) := \alpha(x, z)\alpha(z, x)^{-1}$$

for every 2-cocycle $\alpha \in Z^2(\widetilde{G}, k^\times)$, $x \in \widetilde{G}$, $z \in M_k(G)$. Since Inf is the trivial map, θ_2 is injective.

But because $k^\times$ is abelian and any pairing $f \in P(\widetilde{G}, M_k(G); k^\times)$ defines a group homomorphism $f(-, z)$ for any fixed $z \in M_k(G)$, we obtain that

$$P\big(\widetilde{G}, M_k(G); k^\times\big) \cong P\big(\widetilde{G}/[\widetilde{G}, \widetilde{G}], M_k(G); k^\times\big)\,.$$

Moreover the universal property of the tensor product yields

$$P\big(\widetilde{G}/[\widetilde{G}, \widetilde{G}], M_k(G); k^\times\big) \cong \mathrm{Hom}\,\big(\widetilde{G}/[\widetilde{G}, \widetilde{G}] \otimes_{\mathbb{Z}} M_k(G), k^\times\big).$$

But $\widetilde{G}$ is p'-perfect, i.e. $\widetilde{G}/[\widetilde{G}, \widetilde{G}]$ is a p-group, and $M_k(G)$ is a p'-group. Therefore $\widetilde{G}/[\widetilde{G}, \widetilde{G}] \otimes_{\mathbb{Z}} M_k(G) = \{1\}$. It follows that

$$P(\widetilde{G}, M_k(G); k^\times) = \{1\}$$

and therefore $H^2(\widetilde{G}, k^\times) = \{1\}$ by injectivity of θ_2. ∎

6. Universal p'-central extensions and p'-representation groups

We can now establish the first link between universal p'-central extensions and p'-representation groups.

Lemma 6.1. *If* $1 \longrightarrow M \longrightarrow E \overset{\nu}{\longrightarrow} G \longrightarrow 1$ *is a universal* p'*-central extension of* G*, then* (E, ν) *is a* p'*-representation group of* G*.*

Proof. Let $(\widetilde{G}, \pi_{\widetilde{G}})$ be an arbitrary p'-representation group of G, let V be a finite-dimensional k-vector space, and let $\theta : G \longrightarrow \mathrm{PGL}(V)$ be a group homomorphism. Because $(\widetilde{G}, \pi_{\widetilde{G}})$ has the projective lifting property and (E, ν) is universal, there exist group homomorphisms $\widetilde{\theta} : \widetilde{G} \longrightarrow \mathrm{GL}(V)$ and $\phi : E \longrightarrow \widetilde{G}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M & \longrightarrow & E & \overset{\nu}{\longrightarrow} & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow {\scriptstyle \exists! \phi} & & \| & & \\
1 & \longrightarrow & M_k(G) & \longrightarrow & \widetilde{G} & \overset{\pi_{\widetilde{G}}}{\longrightarrow} & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow {\scriptstyle \exists \widetilde{\theta}} & & \downarrow {\scriptstyle \theta} & & \\
1 & \longrightarrow & k^\times \cdot \mathrm{Id}_V & \longrightarrow & \mathrm{GL}(V) & \overset{\pi_V}{\longrightarrow} & \mathrm{PGL}(V) & \longrightarrow & 1
\end{array}$$

Therefore (E, ν) has the projective lifting property. Now, by Lemma 2.4, both E and G are p'-perfect, that is, $X(E) = \{1\} = X(G)$. Therefore the Hochschild-Serre exact sequence associated to the extension (E, ν) is:

$$1 \longrightarrow 1 \longrightarrow 1 \longrightarrow X(M) \overset{\mathrm{tr}}{\longrightarrow} H^2(G, k^\times) \overset{\mathrm{Inf}}{\longrightarrow} H^2(E, k^\times)$$

Thus the transgression map tr is injective. Therefore, since duality of abelian groups preserves the group order, we have

$$|M| = |\,\mathrm{Hom}(M, k^\times)| = |X(M)| \le |H^2(G, k^\times)| = |M_k(G)|\,.$$

It follows that (E, ν) has the projective lifting property and has a kernel M smaller than $M_k(G)$. By minimality of a p'-representation group, we must have $|M| = |M_k(G)|$, hence $|E| = |\widetilde{G}|$, proving that (E, ν) is a p'-representation group of G. □

We now state our main result. It also provides the second link between universal p'-central extensions and p'-representation groups.

Theorem 6.2. *Let* G *be a finite group.*

(a) *G admits a universal p'-central extension if and only if G is p'-perfect.*
(b) *Moreover, if G is p'-perfect, then a p'-representation group $(\widetilde{G}, \pi_{\widetilde{G}})$ of G is a universal p'-central extension.*

Proof. (a) If G admits a universal p'-central extension, then G is p'-perfect by Lemma 2.4. The converse holds by (b).
(b) By Proposition 5.2, we have $H^2(\widetilde{G}, k^\times) = \{1\}$. By Proposition 5.1, it follows that any p'-central extension of $\widetilde{G}$ splits. Therefore the criterion of Proposition 2.6 implies that $(\widetilde{G}, \pi_{\widetilde{G}})$ is a universal p'-central extension of G. □

Corollary 6.3. *If G is a p'-perfect finite group, then a p'-representation group $(\widetilde{G}, \pi_{\widetilde{G}})$ of G is unique up to isomorphism of group extensions.*

Proof. If $(\widetilde{G}, \pi_{\widetilde{G}})$ and $(\widehat{G}, \pi_{\widehat{G}})$ are two p'-representation groups of G, then by Theorem 6.2(b) they are universal p'-central extensions. Therefore, it follows from both universal properties that there exist two group homomorphisms

$$\phi : \widetilde{G} \longrightarrow \widehat{G} \qquad \text{and} \qquad \psi : \widehat{G} \longrightarrow \widetilde{G}$$

which are inverse to each other and which make the suitable diagrams commute. Thus $(\widetilde{G}, \pi_{\widetilde{G}})$ and $(\widehat{G}, \pi_{\widehat{G}})$ are isomorphic group extensions. □

7. Recognition criteria

To sum up we have proved the following recognition criterion.

Theorem 7.1. *Let $1 \longrightarrow M \longrightarrow E \xrightarrow{\nu} G \longrightarrow 1$ be a p'-central extension. Then (E, ν) is a universal p'-central extension if and only if the following two conditions hold:*

(a) *E is p'-perfect.*

(b) *Every p'-central extension of E splits.*

Proof. Proposition 2.6 asserts that (a) and (b) are sufficient conditions. Conversely, assume that (E, ν) is a universal p'-central extension of G. Then (E, ν) is a p'-representation group of G by Lemma 6.1. Moreover, G is p'-perfect by Lemma 2.4. Thus we have $H^2(E, k^\times) = \{1\}$ by Proposition 5.2. By Proposition 5.1, any p'-central extension of E splits, proving property (b). On the other hand, E is also p'-perfect by Lemma 2.4, so property (a) holds as well. □

Another recognition criterion is the following.

Theorem 7.2. *Let $1 \longrightarrow M \longrightarrow E \xrightarrow{\nu} G \longrightarrow 1$ be a p'-central extension. Then (E, ν) is a universal p'-central extension if and only if the following two conditions hold:*

(a) *E is p'-perfect.*

(b) *(E, ν) has the projective lifting property (relative to k).*

Proof. If (E, ν) is a universal p'-central extension of G, then (a) holds by Lemma 2.4 and (b) holds as well because (E, ν) is a p'-representation group by Lemma 6.1.

Assume now that (a) and (b) hold. Since E is p'-perfect by (a), so is G. Therefore, by Theorem 6.2, G admits a universal p'-central extension of the form

$$1 \longrightarrow M_k(G) \longrightarrow \widetilde{G} \xrightarrow{\pi} G \longrightarrow 1$$

and, by the universal property, there exists a group homomorphism $\phi : \widetilde{G} \longrightarrow E$ making the usual diagram commute:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M_k(G) & \longrightarrow & \widetilde{G} & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi|_{M_k(G)}} & & \downarrow{\scriptstyle \phi} & & \| & & \\ 1 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{\nu} & G & \longrightarrow & 1 \end{array}$$

Let $F = \phi(\widetilde{G})$. Then $\nu(F)$ is the whole of G, so $MF = E$. Since M is central, it follows that F is a normal subgroup of E (because F is normalized by M and by itself). Moreover,

$E/F = MF/F \cong M/(M \cap F)$ is an abelian p'-group because M is. But E is p'-perfect, by assumption (a), so $E/F = \{1\}$ and $F = E$.

Therefore ϕ is surjective, hence $\phi|_{M_k(G)} : M_k(G) \longrightarrow M$ is surjective too and in particular $|M| \le |M_k(G)|$. It follows that the extension (E, ν) has the projective lifting property, by assumption (b), and has a kernel M satisfying $|M| \le |M_k(G)|$. By minimality of a p'-representation group, which has kernel $M_k(G)$, we must have $|M| = |M_k(G)|$. Therefore $\phi|_{M_k(G)} : M_k(G) \longrightarrow M$ is a bijection and it follows that ϕ is an isomorphism, proving that (E, ν) is a universal p'-central extension. □

8. Other universality conditions

The approach used in [1, §11] uses a slightly different notion of universality for central extensions. In this final section, we establish the link with our definition. But before discussing this issue, we show that our definition is equivalent to a property which seems much stronger at first sight.

Lemma 8.1. *If* $1 \longrightarrow M \longrightarrow E \stackrel{\nu}{\longrightarrow} G \longrightarrow 1$ *is a universal* p'*-central extension of a finite group* G*, then for any central extension of groups*

$$1 \longrightarrow B \longrightarrow X \stackrel{\pi}{\longrightarrow} G^* \longrightarrow 1$$

with B *a finite* p'*-group and any group homomorphism* $\theta : G \longrightarrow G^*$*, there exists a unique group homomorphism* $\widetilde{\theta} : E \longrightarrow X$ *such that the following diagram commutes:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M & \longrightarrow & E & \stackrel{\nu}{\longrightarrow} & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \widetilde{\theta}|_M} & & \downarrow{\scriptstyle \exists\widetilde{\theta}} & & \downarrow{\scriptstyle \theta} & & \\
1 & \longrightarrow & B & \longrightarrow & X & \stackrel{\pi}{\longrightarrow} & G^* & \longrightarrow & 1
\end{array}$$

Proof. Let $Y = \pi^{-1}(\theta(G))$ and let $i : Y \longrightarrow X$ be the inclusion map. Then we have a p'-central extension

$$1 \longrightarrow B \longrightarrow Y \stackrel{\pi|_Y}{\longrightarrow} \theta(G) \longrightarrow 1$$

because both B and $\theta(G)$ are finite. Let P be the pull-back of the two maps $\pi|_Y : Y \longrightarrow \theta(G)$ and $\theta : G \longrightarrow \theta(G)$, that is,

$$P = \{\, (y, g) \in Y \times G \mid \pi(y) = \theta(g) \,\},$$

and denote by $pr_Y : P \longrightarrow Y$ and $pr_G : P \longrightarrow G$ the two projections. Then

$$1 \longrightarrow B \longrightarrow P \stackrel{pr_G}{\longrightarrow} G \longrightarrow 1$$

is a central p'-extension of G. By our Definition 2.1, there exists a unique group homomorphism $\phi : E \longrightarrow P$ making the top two rows of the following diagram commute:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{\nu} & G & \longrightarrow & 1 \\
 & & {\scriptstyle \phi|_M}\downarrow & & {\scriptstyle \phi}\downarrow & & \| & & \\
1 & \longrightarrow & B & \longrightarrow & P & \xrightarrow{pr_G} & G & \longrightarrow & 1 \\
 & & \| & & {\scriptstyle pr_Y}\downarrow & & {\scriptstyle \theta}\downarrow & & \\
1 & \longrightarrow & B & \longrightarrow & Y & \xrightarrow{\pi|_Y} & \theta(G) & \longrightarrow & 1 \\
 & & \| & & {\scriptstyle i}\downarrow & & \downarrow & & \\
1 & \longrightarrow & B & \longrightarrow & X & \xrightarrow{\pi} & G^* & \longrightarrow & 1
\end{array}$$

Thus we obtain a map $\widetilde{\theta} := i \circ pr_Y \circ \phi : E \longrightarrow X$ making the right-hand side diagram commute.

In order to show the uniqueness of $\widetilde{\theta}$, let $\psi : E \longrightarrow X$ making the right-hand side diagram commute. Then $\pi \circ \psi(E) = \theta \circ \nu(E) \subseteq \theta(G)$ and therefore $\psi(E) \subseteq \pi^{-1}(\theta(G)) = Y$, so that $\psi = i \circ \psi'$ where $\psi' : E \longrightarrow Y$. We now have two maps $\psi' : E \longrightarrow Y$ and $\nu : E \longrightarrow G$ such that $\pi|_Y \circ \psi' = \theta \circ \nu$. By the universal property of pull-backs, there is a unique group homomorphism $\phi^* : E \longrightarrow P$ such that $pr_Y \circ \phi^* = \psi'$ and $pr_G \circ \phi^* = \nu$. The latter equality implies that $\phi^* = \phi$, by uniqueness of ϕ. It follows that

$$\psi = i \circ \psi' = i \circ pr_Y \circ \phi^* = i \circ pr_Y \circ \phi = \widetilde{\theta}\,,$$

as was to be shown. □

Let us now recall the definition from [1] (see [1, Definition 11.35]).

Definition 8.2. Let B be an abelian group. A central extension

$$1 \longrightarrow M \longrightarrow E \xrightarrow{\nu} G \longrightarrow 1$$

is called B-*universal* if, for any central extension of groups

$$1 \longrightarrow B \longrightarrow X \xrightarrow{\pi} G^* \longrightarrow 1$$

with kernel B and any group homomorphism $\theta : G \longrightarrow G^*$, there exists a group homomorphism $\widetilde{\theta} : E \longrightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{\nu} & G & \longrightarrow & 1 \\
 & & {\scriptstyle \widetilde{\theta}|_M}\downarrow & & {\scriptstyle \exists\,\widetilde{\theta}}\downarrow & & {\scriptstyle \theta}\downarrow & & \\
1 & \longrightarrow & B & \longrightarrow & X & \xrightarrow{\pi} & G^* & \longrightarrow & 1
\end{array}$$

We emphasize that the homomorphism $\widetilde{\theta}$ is not necessarily unique. Therefore, this is not a universal property in the usual categorical sense. However, if E is p'-perfect and B has no p-torsion, then we know that $\widetilde{\theta}$ must be unique, by Lemma 2.5, so we do have a universal property in that case. That is what happens in our next proposition.

The connection with our Definition 2.1 is provided by the following result.

Proposition 8.3. *Let* $1 \longrightarrow M \longrightarrow E \xrightarrow{\nu} G \longrightarrow 1$ *be a* p'*-central extension and assume that* E *is a* p'*-perfect group. The following conditions are equivalent:*

(a) (E,ν) *is* $k^\times$*-universal.*
(b) (E,ν) *is* B*-universal, for every finite* p'*-group* B*.*
(c) (E,ν) *is a universal* p'*-central extension.*

Proof.
(b) $\Rightarrow$ (c): Since E is p'-perfect, this is a straightforward consequence of the definitions together with Lemma 2.5.
(c) $\Rightarrow$ (b): If (E,ν) is a universal p'-central extension, then Lemma 8.1 shows precisely that (E,ν) is B-universal, for every finite p'-group B.
(a) $\Rightarrow$ (c): Assume that (E,ν) is $k^\times$-universal. Since, for any finite-dimensional k-vector space V, the kernel of the central extension

$$1 \longrightarrow k^\times \cdot \mathrm{Id}_V \longrightarrow \mathrm{GL}(V) \longrightarrow \mathrm{PGL}(V) \longrightarrow 1$$

is precisely $k^\times$, we see that (E,ν) has the projective lifting property. But E is p'-perfect by assumption, so Theorem 7.2 applies, showing that (E,ν) is a universal p'-central extension.
(c) $\Rightarrow$ (a): If (E,ν) is a universal p'-central extension, then it is a p'-representation group by Lemma 6.1. In particular, (E,ν) has the projective lifting property (relative to k). In that case, Theorem 11.40 in [1] asserts precisely that (E,ν) is $k^\times$-universal. □

The last argument in the proof uses Theorem 11.40 in [1]. Let us mention that the proof of that result in [1] is based on another characterization of B-universality. Namely, the central extension $1 \longrightarrow M \longrightarrow E \xrightarrow{\nu} G \longrightarrow 1$ is B-universal if and only if the transgression homomorphism $\mathrm{tr} : \mathrm{Hom}(M,B) \longrightarrow H^2(G,B)$ is surjective. We refer to [1, §11] for more details.

Remark 8.4. For the sake of completeness, we note that if G is a perfect group, then many authors define a universal p'-central extension in the following way. In this case, it is a standard result (due to Schur) that G admits a *representation group* relative to $\mathbb{C}$, say $(\widehat{G}, \pi_{\widehat{G}})$, with kernel the Schur multiplier $M(G) := H^2(G,\mathbb{C}^\times)$, which is unique up to isomorphism, and is in fact a *universal central extension.* We refer to [7, Chapter 11] for these results. Then one may define

$$\widetilde{G} := \widehat{G}/M(G)_p$$

where $M(G)_p$ is the p-part of $M(G)$. This induces a central extension

$$1 \longrightarrow M(G)/M(G)_p \longrightarrow \widetilde{G} \xrightarrow{\tilde{\pi}} G \longrightarrow 1$$

where $M(G)/M(G)_p \cong M(G)_{p'} \cong M_k(G)$, which is a universal p'-central extension. We refer to [4, Section 6] for details.

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