SPLENDID MORITA EQUIVALENCES FOR PRINCIPAL 2-BLOCKS WITH DIHEDRAL DEFECT GROUPS

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Abstract. Given a dihedral 2-group P of order at least 8, we classify the splendid Morita equivalence classes of principal 2-blocks with defect groups isomorphic to P. To this end we construct explicit stable equivalences of Morita type induced by specific Scott modules using Brauer indecomposability and gluing methods; we then determine when these stable equivalences are actually Morita equivalences, and hence automatically splendid Morita equivalences. Finally, we compute the generalised decomposition numbers in each case.

1. Introduction

In this paper, we are concerned with the classification of principal 2-blocks with dihedral defect groups of order at least 8, up to *splendid Morita equivalence*, also often called *Puig equivalence*.

This is motivated by a conjecture of Puig's [Pui82] known as Puig's Finiteness Conjecture (see Broué [Bro94, 6.2] or Thévenaz [Thé95, (38.6) Conjecture] for published versions) stating that for a given prime p and a finite p-group P there are only finitely many isomorphism classes of interior P-algebras arising as source algebras of p-blocks of finite groups with defect groups isomorphic to P, or equivalently that there are only a finite number of *splendid Morita equivalence* classes of blocks of finite groups with defect groups isomorphic to P. This obviously strengthens Donovan's Conjecture. However, we emphasise that by contrast to Donovan's Conjecture, if p is a prime number, $(K, \mathcal{O}, k)$ a p-modular system with k algebraically closed, and Puig's Finiteness Conjecture holds over k, then it automatically holds over $\mathcal{O}$, since the bimodules inducing splendid Morita equivalences are liftable from k to $\mathcal{O}$.

The cases where P is either cyclic [Lin96b] or a Klein-four group [CEKL11] are the only cases where this conjecture has been proved to hold in full generality. Else, under additional assumptions, Puig's Finiteness Conjecture has also been proved for several classes of finite groups, as for instance for p-soluble groups [Pui94], for symmetric groups [Pui94], for alternating groups [Kes02], of the double covers thereof [Kes96], for Weyl groups [Kes00], or for classical groups [HK00, HK05, Kes01]. The next cases to investigate should naturally be in tame representation type.

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In this paper, we investigate the principal blocks of groups G with a Sylow 2-subgroup P which is dihedral of order at least 8 over an algebraically closed field k of characteristic 2. Our main result is the following.

Theorem 1.1. *Let $n \geq 3$ be a positive integer. Then the splendid Morita equivalence classes of principal 2-blocks of finite groups with dihedral defect group D_{2^n} of order 2^n coincide with the Morita equivalence classes of such blocks. More accurately, a principal block with dihedral defect group D_{2^n} is splendidly Morita equivalent to precisely one of the following blocks:*

(1) kD_{2^n};
(2) $B_0(k\mathfrak{A}_7)$ *in case* $n = 3$;
(3) $B_0(k[\mathrm{PSL}_2(q)])$, *where q is a fixed odd prime power such that* $(q-1)_2 = 2^n$;
(4) $B_0(k[\mathrm{PSL}_2(q)])$, *where q is a fixed odd prime power such that* $(q+1)_2 = 2^n$;
(5) $B_0(k[\mathrm{PGL}_2(q)])$, *where q is a fixed odd prime power such that* $2(q-1)_2 = 2^n$; *or*
(6) $B_0(k[\mathrm{PGL}_2(q)])$, *where q is a fixed odd prime power such that* $2(q+1)_2 = 2^n$.

In particular, if q and q' are two odd prime powers as in (3)-(6) *such that either* $(q-1)_2 = (q'-1)_2$ *(Cases* (3) *and* (5)*), or* $(q+1)_2 = (q'+1)_2$ *(Cases* (4) *and* (6)*), then* $B_0(k[\mathrm{PSL}_2(q)])$ *is splendidly Morita equivalent to* $B_0(k[\mathrm{PSL}_2(q')])$ *and* $B_0(k[\mathrm{PGL}_2(q)])$ *is splendidly Morita equivalent to* $B_0(k[\mathrm{PGL}_2(q')])$.

Remark 1.2. We note that if G is a soluble group and B is an arbitrary 2-block of G with a defect group $P \cong D_{2^n}$ with $n \geq 3$ which is not nilpotent, then $n = 3$ and B is actually splendidly Morita equivalent to $k\mathfrak{S}_4 \cong k[\mathrm{PGL}_2(3)]$ (see [Kos82]). There is also an interesting and related result by Linckelmann [Lin94] where all derived equivalence classes of blocks B with dihedral defect groups over the field k are determined.

Furthermore, we will prove in Corollary 4.7 that, for a given defect group $P \cong D_{2^n}$ ($n \geq 3$), up to stable equivalence of Morita type, there are exactly three equivalence classes of principal blocks of finite groups G with defect group P, and these depend only on the fusion system $\mathcal{F}_P(G)$, or equivalently on the number of modular simple modules in $B_0(kG)$.

In order to prove Theorem 1.1, we will construct explicit Morita equivalences induced by bimodules given by Scott modules of the form $\mathrm{Sc}(G \times G', \Delta P)$. First we will construct stable equivalences of Morita type using these modules using gluing methods and then determine when these stable equivalences are actually Morita equivalences. To reach this aim, we make use of the notion of Brauer indecomposability, introduced in [KKM11]. In particular, we will use some recent results of Ishioka and Kunugi [IK17] in order to prove the following theorem:

Theorem 1.3. *Let G be a finite group with a dihedral 2-subgroup P of order at least 8. Assume moreover that the fusion system $\mathcal{F}_P(G)$ is saturated and $C_G(Q)$ is 2-nilpotent for every $\mathcal{F}_P(G)$-fully normalised non-trivial subgroup Q of P. Then the Scott module $\mathrm{Sc}(G, P)$ is Brauer indecomposable.*

This result, crucial for our work, may in fact be of independent interest as it is an extension of the main results of [KKL15]. We note that further results on Brauer indecomposability of Scott modules under different hypotheses may be found in [KKM11, Theorem 1.2], [KKL15, Theorem 1.2(b)] and [Tuv14].

The paper is structured as follows. In Section 2 we set up our notation and recall background material which we will use throughout. In Section 3 we establish some properties of Scott modules of direct products with respect to diagonal p-subgroups. From Section 4 onwards, we will assume that the field k has characteristic 2 and we will focus our attention on groups with dihedral Sylow 2-subgroups of order at least 8. In Section 4 we prove Theorem 1.3. In Sections 5 and 6 we determine when the stable equivalences constructed in Section 4 are indeed Morita equivalences. In Section 7 we prove Theorem 1.1, and finally, in Section 8, as a consequence of Theorem 1.1, we can specify the signs occurring in Brauer's computation of the generalised decomposition numbers of principal blocks with dihedral defect groups in [Bra66, §VII]. This will yield the following result:

Corollary 1.4. *If G is a finite group with a dihedral Sylow 2-subgroup of order 2^n with $n \geq 3$, then $|\mathrm{Irr}(B_0(kG))| = 2^{n-2}+3$ and the values at non-trivial 2-elements of the ordinary irreducible characters in $\mathrm{Irr}(B_0(kG))$ are given by the non-trivial generalised decomposition numbers of $B_0(kG)$ and depend only on the splendid Morita equivalence class of $B_0(kG)$.*

Here by *non-trivial* generalised decomposition number, we mean the generalised decomposition numbers parametrised by non-trivial 2-elements.

2. Notation and quoted results

2.1. **Notation.** Throughout this paper, unless otherwise stated we adopt the following notation and conventions. All groups considered are assumed to be finite and all modules over finite group algebras are assumed to be finitely generated unitary right modules. We let G denote a finite group, and k an algebraically closed field of characteristic $p > 0$.

Given a positive integer n, we write $D_{2^n}, C_n, \mathfrak{S}_n$ and $\mathfrak{A}_n$ for the dihedral group of order 2^n, the cyclic group of order n, the symmetric group of degree n and the alternating group of degree n, respectively. We write $H \leq G$ when H is a subgroup of G. Given two finite groups N and H, we denote by $N \rtimes H$ a semi-direct product of N by H (where $N \lhd (N \rtimes H)$). For a subset S of G, we set $S^g := g^{-1}Sg$, and for $h \in G$ we set $h^g := g^{-1}hg$. For an integer $n \geq 1$, $\langle g_1, \dots, g_n \rangle$ is the subgroup of G generated by the elements $g_1, \dots, g_n \in G$. We denote the centre of G by $Z(G)$ and we set $\Delta G := \{(g,g) \in G \times G \,|\, g \in G\} \leq G \times G$.

Given a p-subgroup $P \leq G$ we denote by $\mathcal{F}_P(G)$ the fusion system of G on P; that is the category whose objects are the p-subgroups of P, and whose morphisms from Q to R are the group homomorphisms induced by conjugation by elements of G, see [AKO11, Definition I.2.1]. We recall that if P is a Sylow p-subgroup of G, then $\mathcal{F}_P(G)$ is saturated, see [AKO11, Definition I.2.2]. For further notation and terminology on fusion systems, we refer to [AKO11] and [BLO03].

The trivial kG-module is denoted by k_G. If $H \leq G$ is a subgroup, M is a kG-module and N is a kH-module, then we write $M^* := \mathrm{Hom}_k(M,k)$ for the k-dual of M, $M{\downarrow}_H$ for the restriction of M to H and $N{\uparrow}^G$ for the induction of N to G. Given an $H \leq G$, we denote by $P_H(k_G)$ the H-projective cover of the trivial module k_G and we let $\Omega_H(k_G)$ denote the H-relative Heller operator, that is $\Omega_H(k_G) = \mathrm{Ker}\,(P_H(k_G) \twoheadrightarrow k_G)$, the kernel of the canonical projection (see [Thé85]). We write $B_0(kG)$ for the principal block of kG. For a p-block B, we denote by $\mathrm{Irr}(B)$ the set of ordinary irreducible characters in B and by $\mathrm{IBr}(B)$ the set of irreducible Brauer characters in B. Further we use the standard notation $k(B) := |\mathrm{Irr}(B)|$ and $l(B) := |\mathrm{IBr}(B)|$.

For a subgroup $H \leq G$ we denote the (Alperin-)Scott kG-module with respect to H by $\mathrm{Sc}(G,H)$. By definition $\mathrm{Sc}(G,H)$ is the unique indecomposable direct summand of the induced module $k_H{\uparrow}^G$ which contains k_G in its top (or equivalently in its socle). If $Q \in \mathrm{Syl}_p(H)$, then Q is a vertex of $\mathrm{Sc}(G,H)$ and a p-subgroup of G is a vertex of $\mathrm{Sc}(G,H)$ if and only if it is G-conjugate to Q. It follows that $\mathrm{Sc}(G,H) = \mathrm{Sc}(G,Q)$. We refer the reader to [Bro85, §2] and [NT88, Chap.4 §8.4] for these results. Furthermore, we will need the fact that $\mathrm{Sc}(G,H)$ is nothing else but the relative H-projective cover $P_H(k_G)$ of the trivial module k_G; see [Thé85, Proposition 3.1]. In order to produce splendid Morita equivalences between principal blocks of two finite groups G and G' with a common defect group P, we mainly use Scott modules of the form $\mathrm{Sc}(G \times G', \Delta P)$, which are obviously $(B_0(kG), B_0(kG'))$-bimodules by the previous remark.

For further notation and terminology, we refer the reader to the books [Gor68], [NT88] and [Thé95].

2.2. **Equivalences of block algebras.** Let G and H be two finite groups, and let A and B be block algebras of kG and kH with defect groups P and Q, respectively.

The algebras A and B are called *splendidly Morita equivalent* (or *Puig equivalent*), if there is a Morita equivalence between A and B induced by an (A,B)-bimodule M such that M, seen as a right $k[G \times H]$-module, is a p-permutation module. In this case, we write $A \sim_{SM} B$. Due to a result of Puig (see [Pui99, Corollary 7.4] and [Lin18, Proposition 9.7.1]), the defect groups P and Q are isomorphic (and hence from now on we identify P and Q). Obviously M is indecomposable as a $k(G \times H)$-module. Further since ${}_AM$ and M_B are both projective, M has a vertex R which is written as $R = \Delta(P) \leq G \times H$. Then, this is equivalent to the condition that A and B have source algebras which are isomorphic as interior P-algebras by the result of Puig and Scott (see [Lin01, Theorem 4.1] and [Pui99, Remark 7.5]).

In particular, we note that if the Scott module $M := \mathrm{Sc}(G \times H, \Delta P)$ induces a Morita equivalence between the principal blocks A and B of kG and kH, respectively, then this is a splendid Morita equivalence because Scott modules are p-permutation modules by definition.

Let now M be an (A,B)-bimodule and N a (B,A)-bimodule. Following Broué [Bro94, §5], we say that the pair (M,N) induces a *stable equivalence of Morita type* between A and B if M and N are projective both as left and right modules, and there are isomorphisms of (A,A)-bimodules and (B,B)-bimodules

$$M \otimes_B N \cong A \oplus X \qquad \text{and} \qquad N \otimes_A M \cong B \oplus Y\,,$$

respectively, where X is a projective (A,A)-bimodule and Y is a projective (B,B)-bimodule.

The following result of Linckelmann will allow us to construct Morita equivalences using stable equivalences of Morita type.

Theorem 2.1 ([Lin96, Theorem 2.1(ii),(iii)]). *Let A and B be finite-dimensional k-algebras which are indecomposable non-simple self-injective k-algebras. Let M be an (A,B)-bimodule inducing a stable equivalence between A and B.*

(a) *If M is indecomposable, then for any simple A-module S, the B-module $S \otimes_A M$ is indecomposable and non-projective.*
(b) *If for any simple A-module S, the B-module $S \otimes_A M$ is simple, then the functor $- \otimes_A M$ induces a Morita equivalence between A and B.*

Furthermore, we recall the following fundamental result, originally due to Alperin [Alp76] and Dade [Dad77], which will provide us with an important source of splendid Morita equivalences in Section 6.

Theorem 2.2 ([KK02, (3.1) Lemma]). *Let $\widetilde{G}$ be a finite group. Let $G \trianglelefteq \widetilde{G}$ be a normal subgroup such that $\widetilde{G}/G$ is a p'-group and $\widetilde{G} = G\,C_{\widetilde{G}}(P)$, where P is a Sylow p-subgroup of G. Furthermore, let $\tilde{e}$ and e be the block idempotents corresponding to $B_0(k\widetilde{G})$ and $B_0(kG)$, respectively. Then the following holds:*

(a) *The map $B_0(kG) \longrightarrow B_0(k\widetilde{G}), a \mapsto a\tilde{e}$ is a k-algebra isomorphism, so that $e\tilde{e} = \tilde{e}e = \tilde{e}$.*
(b) *The right $k[\widetilde{G} \times G]$-module $B_0(k\widetilde{G}) = \tilde{e}k\widetilde{G} = \tilde{e}k\widetilde{G}e$ induces a splendid Morita equivalence between $B_0(k\widetilde{G})$ und $B_0(kG)$.*

2.3. **The Brauer construction and Brauer indecomposability.** Given a kG-module V and a p-subgroup $Q \leq G$, the *Brauer construction* (or *Brauer quotient*) of V with respect to Q is defined to be the $kN_G(Q)$-module

$$V(Q) := V^Q / \sum_{R<Q} \mathrm{Tr}_R^Q(V^R)\,,$$

where V^Q denotes the set of Q-fixed points of V, and for each proper subgroup $R < Q$, $\mathrm{Tr}_R^Q : V^R \longrightarrow V^Q, v \mapsto \sum_{xR \in Q/R} xv$ denotes the relative trace map. See e.g. [Thé95, §27]. We recall that the Brauer construction with respect to Q sends a p-permutation kG-module V functorially to the p-permutation $kN_G(Q)$-module $V(Q)$, see [Bro85, p.402].

Furthermore, following the terminology introduced in [KKM11], a kG-module V is said to be *Brauer indecomposable* if the $kC_G(Q)$-module $V(Q)\downarrow_{C_G(Q)}^{N_G(Q)}$ is indecomposable or zero for each p-subgroup $Q \leq G$.

In order to detect Brauer indecomposability, we will use the following two recent results of Ishioka and Kunugi:

Theorem 2.3 ([IK17, Theorem 1.3]). *Let G be a finite group and P a p-subgroup of G. Let $M := \mathrm{Sc}(G,P)$. Assume that the fusion system $\mathcal{F}_P(G)$ is saturated. Then the following assertions are equivalent:*

(i) *M is Brauer indecomposable.*
(ii) *$\mathrm{Sc}(N_G(Q), N_P(Q))\downarrow_{QC_G(Q)}^{N_G(Q)}$ is indecomposable for each $\mathcal{F}_P(G)$-fully normalised subgroup Q of P.*

Theorem 2.4 ([IK17, Theorem 1.4]). *Let G be a finite group and P a p-subgroup of G, Q an $\mathcal{F}_P(G)$-fully normalised subgroup of P, and suppose that $\mathcal{F}_P(G)$ is saturated. Assume moreover that there exists a subgroup H_Q of $N_G(Q)$ satisfying the following two conditions:*

(1) *$N_P(Q) \in \mathrm{Syl}_p(H_Q)$; and*
(2) *$|N_G(Q) : H_Q| = p^a$ $(a \geq 0)$.*

Then $\mathrm{Sc}(N_G(Q), N_P(Q))\downarrow_{QC_G(Q)}^{N_G(Q)}$ is indecomposable.

2.4. **Principal blocks with dihedral defect groups.** Since $O_{2'}(G)$ acts trivially on the principal block of G it is well-known that $B_0(G)$ and $B_0(G/O_{2'}(G))$ are Morita equivalent. Such a Morita equivalence is induced by the $(B_0(k[G/O_{2'}(G)]), B_0(kG))$-bimodule $B_0(k[G/O_{2'}(G)])$, which is obviously a 2-permutation module. Hence these blocks are indeed splendidly Morita equivalent, and we may restrict our attention to the case $O_{2'}(G) = \{1\}$.

We recall that if G is a finite group with a dihedral Sylow 2-subgroup P of order at least 8, then Gorenstein and Walter [GW65] proved that $G/O_{2'}(G)$ is isomorphic to either

(D1) P,
(D2) the alternating group $\mathfrak{A}_7$, or
(D3) a subgroup of $\mathrm{P\Gamma L}_2(q)$ containing $\mathrm{PSL}_2(q)$, where q is a power of an odd prime. In other words, one of the following groups:
   (i) $\mathrm{PSL}_2(q) \rtimes C_f$ where q is a power of an odd prime such that $q \equiv \pm 1 \pmod 8$, and $f \geq 1$ is a suitable odd number; or
   (ii) $\mathrm{PGL}_2(q) \rtimes C_f$ where q is a power of an odd prime and $f \geq 1$ is a suitable odd number.

The fact that q is a power of an odd prime can be found in [Suz86, Chapter 6 (8,9)]. Moreover, the splitting of case (D3) into (i) and (ii) follows from the fact that

$$\mathrm{P\Gamma L}_2(q) \cong \mathrm{PGL}_2(q) \rtimes \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_r)\,,$$

where $q = r^m$ is a power of an odd prime r and the Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_r)$ is cyclic of order m, generated by the Frobenius automorphism $F : \mathbb{F}_q \longrightarrow \mathbb{F}_q, x \mapsto x^r$. This implies that C_f is a cyclic subgroup of $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_r)$ generated by a power of F, and moreover the requirement that P is dihedral forces f to be odd. (Here we apply [Suz86, Chapter 6 (8.9)] implying that C_f is of odd order. See also the beginning of [Gor68, §16.3].)

2.5. **Dihedral 2-groups and 2-fusion.** Assume G is a finite group having a Sylow 2-subgroup P which is a dihedral 2-group D_{2^n} of order 2^n $(n \geq 3)$. Write

$$P := D_{2^n} := \langle s, t \mid s^{2^{n-1}} = t^2 = 1, tst = s^{-1} \rangle$$

and set $z := s^{2^{n-2}}$, so that $\langle z \rangle = Z(P)$. Then there are three possible fusion systems $\mathcal{F}_P(G)$ on P.

(1) First case: $\mathcal{F}_P(G) = \mathcal{F}_P(P)$. There are exactly three G-conjugacy classes of involutions in P: $\{z\}$, $\{s^{2j}t \mid 0 \leq j \leq 2^{n-2}-1\}$ and $\{s^{2j+1}t \mid 0 \leq j \leq 2^{n-2}-1\}$. Moreover, $l(B_0(kG)) = 1$, that is $B_0(kG)$ possesses exactly one simple module, namely the trivial module k_G.
(2) Second case: $\mathcal{F}_P(G) = \mathcal{F}_P(\mathrm{PGL}_2(q))$, where $2(q \pm 1)_2 = 2^n$. There are exactly two G-conjugacy classes of involutions in P, represented by the elements z and st. Note that t is fused with z in this case. Moreover, $l(B_0(kG)) = 2$.
(3) Third case: $\mathcal{F}_P(G) = \mathcal{F}_P(\mathrm{PSL}_2(q))$, where $(q \pm 1)_2 = 2^n$. There is exactly one G-conjugacy class of involutions in P, represented by z. Moreover, $l(B_0(kG)) = 3$.

In fact, if P is a 2-subgroup of G, but not necessarily a Sylow 2-subgroup, and $\mathcal{F}_P(G)$ is saturated, then $\mathcal{F}_P(G)$ is isomorphic to one of the fusion systems in (1), (2), and (3). We refer the reader to [Gor68, §7.7], [Bra66, §VII] and [CG12, Theorem 5.3] for these results.

Lemma 2.5. *Let $G := \mathrm{PGL}_2(q)$ for a prime power q such that the Sylow 2-subgroups of G are dihedral of order at least 8, and let $H \leq G$ be a subgroup isomorphic to $\mathrm{PSL}_2(q)$. Moreover, let $Q \in \mathrm{Syl}_2(H)$ and $P \in \mathrm{Syl}_2(G)$ such that $P \cap H = Q$. Without loss of generality we may set $P := \langle s, t \mid s^{2^{n-1}} = t^2 = 1, tst = s^{-1}\rangle$ and $Q := \langle s^2, t\rangle$. Then the following holds:*

(a) *st is an involution in $P \setminus Q$, and moreover, any two involutions in $P \setminus Q$ are P-conjugate.*

(b) *Set $z := s^{2^{n-2}}$. Then centralisers of involutions in P and G are given as follows:*
   (i) *$C_P(z) = P$, $C_P(t) = \langle t, z\rangle \cong C_2 \times C_2$ and $C_P(st) = \langle st, z\rangle \cong C_2 \times C_2$.*
   (ii) *If $q \equiv 1 \pmod 4$, then $C_G(t) \cong D_{2(q-1)}$ and $C_G(st) \cong D_{2(q+1)}$.*
   (iii) *If $q \equiv -1 \pmod 4$, then $C_G(t) \cong D_{2(q+1)}$ and $C_G(st) \cong D_{2(q-1)}$.*
   In particular $C_P(z) \in \mathrm{Syl}_2(C_G(z))$ and $C_P(st) \in \mathrm{Syl}_2(C_G(st))$.

Proof. By assumption $|G : H| = 2$, $P \cong D_{2^n}$ with $n \geq 3$ and $Q \cong D_{2^{n-1}}$. The three P-conjugacy classes of involutions in P are

$$\{s^{2^{n-2}}\}, \{s^{2j}t \mid 0 \leq j \leq 2^{n-2}\} \text{ and } \{s^{2j+1}t \mid 0 \leq j \leq 2^{n-2}\},$$

where $\{s^{2^{n-2}}\}, \{s^{2j}t \mid 0 \leq j \leq 2^{n-2}\} \subset Q$ and $\{s^{2j+1}t \mid 0 \leq j \leq 2^{n-2}\} \subset P \setminus Q$. Part (a) follows.

For part (b), (i) is obvious and for (ii) and (iii) we refer to [GW62, §4(B)] for the description of centralisers of involutions in $\mathrm{PGL}_2(q)$. Now it is clear that $C_P(z) \in \mathrm{Syl}_2(C_G(z))$. If $q \equiv 1 \pmod 4$, then $2 \mid\mid q+1$, so that $|C_G(st)|_2 = |D_{2(q+1)}|_2 = 4 = |C_P(st)|$, whereas if $q \equiv -1 \pmod 4$, then $2 \mid\mid q-1$, so that $|C_G(st)|_2 = |D_{2(q-1)}|_2 = 4 = |C_P(st)|$. Hence $C_P(st) \in \mathrm{Syl}_2(C_G(st))$. □

3. Properties of Scott modules

Lemma 3.1. *Let G and G' be finite p-nilpotent groups with a common Sylow p-subgroup P. Then $\mathrm{Sc}(G \times G', \Delta P)$ induces a Morita equivalence between $B_0(kG)$ and $B_0(kG')$.*

Proof. Set $M := \mathrm{Sc}(G \times G', \Delta P)$, $B := B_0(kG)$ and $B' := B_0(kG')$. By definition, we have

$$M \,\Big|\, 1_B \cdot kG \otimes_{kP} kG' \cdot 1_{B'}\,.$$

Since G and G' are p-nilpotent $1_B \cdot kG \otimes_{kP} kG' \cdot 1_{B'} = kP \otimes_{kP} kP \cong kP$ as (kG, kG')-bimodules. Now as kP is indecomposable as a (kP, kP)-bimodule, it is also indecomposable as a (kG, kG')-bimodule. Therefore $M \cong kP$ as (B, B')-bimodule. Therefore M induces a Morita equivalence between B and B'. □

Lemma 3.2. *Assume that $p = 2$. Let G and G' be two finite groups with a common Sylow 2-subgroup P, and assume that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Let z be an involution in $Z(P)$. Then*

$$\mathrm{Sc}\Big(C_G(z) \times C_{G'}(z),\, \Delta P\Big) \,\Big|\, \Big(\mathrm{Sc}(G \times G',\, \Delta P)\Big)(\Delta\langle z\rangle).$$

Proof. Since $\mathcal{F}_P(G) = \mathcal{F}_P(G')$, we have $\mathcal{F}_{\Delta P}(G \times G') \cong \mathcal{F}_P(G)$. Since P is a Sylow 2-subgroup of G, $\mathcal{F}_P(G)$ is saturated, and therefore $\mathcal{F}_{\Delta P}(G \times G')$ is also saturated. Since $\Delta\langle z\rangle \leq Z(\Delta P)$, $\Delta\langle z\rangle$ is a fully normalised subgroup of ΔP with respect to $\mathcal{F}_{\Delta P}(G \times G')$ by definition. Thus it follows from [IK17, Lemmas 3.1 and 2.2] that

$$\mathrm{Sc}(N_{G\times G'}(\Delta\langle z\rangle),\, N_{\Delta P}(\Delta\langle z\rangle)) \,\Big|\, \mathrm{Sc}(G \times G',\, \Delta P){\downarrow}_{N_{G\times G'}(\Delta\langle z\rangle)}.$$

Since z is an involution in $Z(P)$, the above reads

$$\mathrm{Sc}(C_G(z)\times C_{G'}(z),\,\Delta P)\,\Big|\,\mathrm{Sc}(G\times G',\,\Delta P){\downarrow}_{C_G(z)\times C_{G'}(z)}.$$

Set $M := \mathrm{Sc}(G\times G',\,\Delta P)$ and $\mathfrak{M} := \mathrm{Sc}(C_G(z)\times C_{G'}(z),\,\Delta P)$. Since taking the Brauer construction is functorial, by the above we have that

$$\mathfrak{M}(\Delta\langle z\rangle)\,\Big|\,M(\Delta\langle z\rangle).$$

Obviously $\Delta\langle z\rangle$ is in the kernel of the $(kC_G(z)\times kC_{G'}(z))$-bimodule $kC_G(z)\otimes_{kP}kC_{G'}(z)$, and hence it is in the kernel of $\mathfrak{M}$, so that $\mathfrak{M}^{\Delta\langle z\rangle}=\mathfrak{M}$. Therefore by definition of the Brauer construction, we have $\mathfrak{M}(\Delta\langle z\rangle)=\mathfrak{M}$. Therefore $\mathfrak{M}\mid M(\Delta\langle z\rangle)$. $\square$

The following is well-known, but does not seem to appear in the literature. For completeness we provide a proof, which is due to N. Kunugi.

Lemma 3.3. *Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G)=\mathcal{F}_P(G')$. Assume further that M is an indecomposable ΔP-projective p-permutation $k(G\times G')$-module. Let Q be a subgroup of P. Then the following are equivalent.*

(i) $\mathrm{Sc}(G',Q)\mid(k_G\otimes_{kG}M)$.

(ii) $\mathrm{Sc}(G\times G',\,\Delta Q)\mid M$.

Proof. By assumption $M\mid k_{\Delta R}{\uparrow}^{G\times G'}$, where ΔR is a vertex of M with $R\le P$. Set $N := k_{\Delta Q}{\uparrow}^{G\times G'} = kG\otimes_{kQ}kG'$. Thus $N=\mathrm{Sc}(G\times G',\,\Delta Q)\oplus N_0$ where N_0 is a $k(G\times G')$-module which satisfies $\mathrm{Hom}_{k(G\times G')}(N_0,\,k_{G\times G'})=0$, by definition of a Scott module. We have that

$$\begin{aligned} k_G\otimes_{kG}N &= k_G\otimes_{kG}(kG\otimes_{kQ}kG') \\ &= k_G{\downarrow}_Q{\uparrow}^{G'} = k_Q{\uparrow}^{G'} \\ &= \mathrm{Sc}(G',Q)\oplus X\,, \end{aligned}$$

where X is a kG'-module such that no Scott module can occur as a direct summand of X thanks to Frobenius Reciprocity. Now, let us consider an arbitrary $k(G\times G')$-module L with $L\mid N$. Then,

$$\begin{aligned} \dim[\mathrm{Hom}_{kG'}(k_G\otimes_{kG}L,\,k_{G'})] &= \dim[\mathrm{Hom}_{k(G\times G')}(L,\,k_G\otimes_k k_{G'})] \\ &= \dim[\mathrm{Hom}_{k(G\times G')}(L,\,k_{G\times G'})] \\ &\le \dim[\mathrm{Hom}_{G\times G'}(N,\,k_{G\times G'})]=1 \end{aligned}$$

by adjointness and Frobenius Reciprocity. Hence we have

$$\dim[\mathrm{Hom}_{k(G\times G')}(L,\,k_{G\times G'})]=\begin{cases}1 & \text{if } \mathrm{Sc}(G\times G',\,\Delta Q)\mid L\\ 0 & \text{otherwise.}\end{cases}$$

Namely,

$$\mathrm{Sc}(G',Q)\mid(k_G\otimes_{kG}L)\quad\text{if and only if}\quad\mathrm{Sc}(G\times G',\,\Delta Q)\mid L\,.$$

Now, we can decompose M into a direct sum of indecomposable $k(G\times G')$-modules

$$M=M_1\oplus\cdots\oplus M_s\oplus Y_1\oplus\cdots\oplus Y_t$$

for integers $s \geq 1$ and $t \geq 0$, and where for each $1 \leq i \leq s$, $M_i = \mathrm{Sc}(G \times G', \Delta Q_i)$ for some $Q_i \leq P$, and for each $1 \leq j \leq t$, $Y_j \ncong \mathrm{Sc}(G \times G', \Delta S)$ for any $S \leq P$. Then for each $1 \leq i \leq s$, we have

$$k_G \otimes_{kG} M_i = k_G \otimes_{kG} \mathrm{Sc}(G \times G', \Delta Q_i) \,\Big|\, k_G \otimes_{kG} (kG \otimes_{kQ_i} kG')$$

where

$$k_G \otimes_{kG} (kG \otimes_{kQ_i} kG') = k_G{\downarrow_{Q_i}}{\uparrow^{G'}} = k_{Q_i}{\uparrow^{G'}} = \mathrm{Sc}(G', Q_i) \oplus W_i$$

for a kG'-module W_i such that $\mathrm{Hom}_{kG'}(k_{G'}, W_i) = 0$ by Frobenius Reciprocity. Now $\mathcal{F}_P(G) = \mathcal{F}_P(G')$ implies that $\mathcal{F}_P(G) = \mathcal{F}_P(G') \cong \mathcal{F}_{\Delta P}(G \times G')$. This means that for $1 \leq j, j' \leq s$, we have that Q_j is G'-conjugate to $Q_{j'}$ if and only if ΔQ_j is $(G \times G')$-conjugate to $\Delta Q_{j'}$. Therefore the claim follows directly from the characterisation of Scott modules in [NT88, Corollary 4.8.5]. □

Lemma 3.4. *Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Set $M := \mathrm{Sc}(G \times G', \Delta P)$, $B := B_0(kG)$ and $B' := B_0(kG')$. If M induces a stable equivalence of Morita type between B and B', then the following holds:*

(a) $k_G \otimes_B M = k_{G'}$.
(b) *If U is an indecomposable p-permutation kG-module in B with vertex $1 \neq Q \leq P$, then $U \otimes_B M$ has, up to isomorphism, a unique indecomposable direct summand V with vertex Q and V is again a p-permutation module.*
(c) *For any $Q \leq P$,* $\mathrm{Sc}(G, Q) \otimes_B M = \mathrm{Sc}(G', Q) \oplus (\mathrm{proj})$.
(d) *For any $Q \leq P$,* $\Omega_Q(k_G) \otimes_B M = \Omega_Q(k_{G'}) \oplus (\mathrm{proj})$.

Proof. (a) Apply Lemma 3.3 to the case that $Q = P$. Then Condition (ii) is trivially satisfied. Thus we have $k_{G'} \,|\, (k_G \otimes_B M)$, since $\mathrm{Sc}(G', P) = k_{G'}$. Hence Theorem 2.1(a) yields $k_{G'} = k_G \otimes_B M$.

(b) Let U be an indecomposable p-permutation kG-module with vertex Q. Since M induces a stable equivalence of Morita type between B and B', $U \otimes_B M$ has a unique non-projective indecomposable direct summand, say V. Then

$$\begin{aligned} V \,\Big|\, (U \otimes_{kG} M) \,\Big|\, &[U \otimes_{kG} (kG \otimes_{kP} kG')] \\ &= U{\downarrow_P}{\uparrow^{G'}} \,\Big|\, k_Q{\uparrow^G}{\downarrow_P}{\uparrow^{G'}} = \bigoplus_{g \in [Q\backslash G/P]} k_{Q^g \cap P}{\uparrow^{G'}} \end{aligned}$$

by the Mackey decomposition. Hence V is a p-permutation kG'-module which is $(Q^g \cap P)$-projective for an element $g \in G$. Thus there is a vertex R of V with $R \leq Q^g \cap P$. This means that $gRg^{-1} \leq Q \cap gPg^{-1} \leq Q \leq P$. Then, since $\mathcal{F}_P(G) = \mathcal{F}_P(G')$, there is an element $g' \in G'$ such that $grg^{-1} = g'rg'^{-1}$ for every $r \in R$. Hence $g'Rg'^{-1}$ is also a vertex of V and $g'Rg'^{-1} = gRg^{-1} \leq Q$, and hence $R \leq_{G'} Q$.

Similarly, we obtain that $Q \leq_G R$ since M^* induces a stable equivalence of Morita type between B' and B. This implies that $R =_{G'} Q$ and $R =_G Q$.

(c) Set $F := - \otimes_B M$, the functor inducing the stable equivalence of Morita type between B and B'. Fix Q with $1 \neq Q \leq P$, and set $U_Q := \mathrm{Sc}(G, Q)$. Then, we have

$$\begin{aligned} 1 &= \dim[\,\mathrm{Hom}_B(U_Q,\, k_G)] \\ &= \dim[\,\underline{\mathrm{Hom}}_B(U_Q,\, k_G)] \\ &= \dim[\,\underline{\mathrm{Hom}}_{B'}\Big(F(U_Q),\, F(k_G)\Big)] \\ &= \dim[\,\underline{\mathrm{Hom}}_{B'}\Big(F(U_Q),\, k_{G'}\Big)] \\ &= \dim[\,\mathrm{Hom}_{B'}\Big(F(U_Q),\, k_{G'}\Big)]\,, \end{aligned}$$

where the second equality holds because k_G is simple, and the last but one equality holds by (a). Let V_Q be the unique (up to isomorphism) non-projective indecomposable direct summand of $F(U_Q)$. By (b), we know that Q is a vertex of V_Q. Moreover, by the above, we have that

$$\dim[\mathrm{Hom}_{B'}(V_Q,\, k_{G'})] \leq 1\,.$$

We claim that in fact equality holds. Indeed, if $\dim[\mathrm{Hom}_{B'}(V_Q,\, k_{G'})] = 0$, then the above argument implies that

$$\dim[\mathrm{Hom}_{kG}(F^{-1}(V_Q),\, k_G)] = 0$$

as well, but this is a contradiction since U_Q is a direct summand of $F^{-1}(V_Q)$, as the functor F gives a stable equivalence between B and B'. Hence the dimension is one, and we conclude that $V_Q = \mathrm{Sc}(G', Q)$.

Next assume that $Q = 1$, so that $\mathrm{Sc}(G, Q) = \mathrm{Sc}(G, 1) = P(k_G)$. Since

$$P(k_G) \otimes_{kG} M \,\Big|\, P(k_G) \otimes_{kG} kG \otimes_{kP} kG' = P(k_G){\downarrow_P}{\uparrow^{G'}}$$

by definition of a Scott module, $P(k_G) \otimes_{kG} M$ is a projective kG'-module. Moreover, it follows from the adjointness and (a) that

$$\dim[\mathrm{Hom}_{kG'}(P(k_G) \otimes_{kG} M,\, k_{G'})] = \dim[\mathrm{Hom}_{kG}(P(k_G),\, k_{G'} \otimes_{kG'} M^*)] = 1$$

and hence $P(k_G) \otimes_{kG} M = P(k_{G'}) \oplus \mathcal{P}$ where $\mathcal{P}$ is a projective B'-module that does not have $P(k_{G'})$ as a direct summand.

(d) Recall that $\mathrm{Sc}(G, Q) = P_Q(k_G)$, $\mathrm{Sc}(G', Q) = P_Q(k_{G'})$, and $\Omega_Q(k_G) = P_Q(k_G)/k_G$ and $\Omega_Q(k_{G'}) = P_Q(k_{G'})/k_{G'}$. Therefore (d) follows directly from (a), (c) and the stripping-off method [KMN11, (A.1) Lemma]. □

4. Constructing stable equivalences of Morita type

We start with the following gluing result which will allow us to construct stable equivalences of Morita type. It is essentially due to Broué ([Bro94, 6.3.Theorem]), Rouquier ([Rou01, Theorem 5.6]) and Linckelmann ([Lin01, Theorem 3.1]). We aim at using equivalence (iii), which slightly generalises the statement of [Rou01, Theorem 5.6]. Since a statement under our hypotheses does not seem to appear in the literature, we give a proof for completeness.

Lemma 4.1. *Let G and G' be finite groups with a common Sylow p-subgroup P, and assume that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Set $M := \mathrm{Sc}(G \times G', \Delta P)$, $B := B_0(kG)$ and $B' := B_0(kG')$. Further, for each subgroup $Q \leq P$ we set $B_Q := B_0(kC_G(Q))$ and $B'_Q := B_0(kC_{G'}(Q))$. Then, the following three conditions are equivalent.*

(i) *The pair $(M,\, M^*)$ induces a stable equivalence of Morita type between B and B'.*

(ii) *For every non-trivial subgroup* $Q \leq P$*, the pair* $(M(\Delta Q), M(\Delta Q)^*)$ *induces a Morita equivalence between* B_Q *and* B'_Q*.*

(iii) *For every cyclic subgroup* $Q \leq P$ *of order* p*, the pair* $(M(\Delta Q), M(\Delta Q)^*)$ *induces a Morita equivalence between* B_Q *and* B'_Q*.*

Proof. (i)⇔(ii) is a special case of [Lin01, Theorem 3.1], and (ii)⇒(iii) is trivial.

(iii)⇒(ii): Take an arbitrary non-trivial subgroup $Q \leq P$. Then there is a normal series

$$C_p \cong Q_1 \lhd Q_2 \lhd \cdots \lhd Q_m = Q$$

for an integer $m \geq 1$. We shall prove (ii) by induction on m. If $m = 1$, then $M(\Delta Q)$ induces a Morita equivalence between B_Q and B'_Q by (iii).

Next assume that $m \geq 2$ and (ii) holds for $m - 1$, and set $R := Q_{m-1}$. Namely, by the inductive hypothesis $M(\Delta R)$ realises a Morita equivalence between B_R and B'_R. That is to say, we have

$$B_R \cong M(\Delta R) \otimes_{B'_R} M(\Delta R)^*. \tag{1}$$

Moreover,

$$B_R(\Delta Q) = B_Q \tag{2}$$

since $(kC_G(R))(\Delta Q) = kC_G(Q)$ (note that since $R \lhd Q$, $kC_G(R)$ is a right $k\Delta Q$-module). Further, it follows from [Ric96, proof of Theorem 4.1] that

$$\Big(M(\Delta R) \otimes_{B'_R} M(\Delta R)^*\Big)(\Delta Q) \;=\; (M(\Delta R))\,(\Delta Q) \otimes_{B'_Q} [(M(\Delta R))\,(\Delta Q)]^*. \tag{3}$$

Recall that by [BP80, Proposition 1.5] we have

$$(M(\Delta R))\,(\Delta Q) = M(\Delta Q). \tag{4}$$

Hence

$$\begin{aligned} B_Q &= (B_R)(\Delta Q) \quad \text{by (2)} \\ &= \Big(M(\Delta R) \otimes_{B'_R} M(\Delta R)^*\Big)(\Delta Q) \quad \text{by (1)} \\ &= (M(\Delta R))\,(\Delta Q) \otimes_{B'_Q} [(M(\Delta R))\,(\Delta Q)]^* \quad \text{by (3)} \\ &= M(\Delta Q) \otimes_{B'_Q} M(\Delta Q)^* \quad \text{by (4)}\,. \end{aligned}$$

Thus, by making use of [Ric96, Theorem 2.1], we obtain that the pair $(M(\Delta Q), M(\Delta Q)^*)$ induces a Morita equivalence between B_Q and B'_Q. □

From now on and until the end of this article we assume that k has characteristic 2.

The following is an easy application of the Baer-Suzuki theorem, which is essential to treat dihedral defect groups.

Lemma 4.2. *Let* G *be a finite group and let* Q *be a normal* 2*-subgroup of* G *such that* $G/Q \cong \mathfrak{S}_3$*. Assume further that there is an involution* $t \in G \setminus Q$*. Then* G *has a subgroup* H *such that* $t \in H \cong \mathfrak{S}_3$*.*

Proof. Obviously $Q = O_2(G)$ since $G/Q \cong \mathfrak{S}_3$. Therefore, by the Baer-Suzuki theorem (see [Gor68, Theorem 3.8.2]), there exists an element $y \in G$ such that y is conjugate to t in G and the group $\widetilde{H} := \langle t, y\rangle$ is not a 2-group. Therefore $6 \mid |\widetilde{H}|$, and since $\widetilde{H}$ is generated by two involutions, it is a dihedral group of order $3 \cdot 2^a$ for some positive integer a, that is $\widetilde{H} = C_{3\cdot 2^{a-1}} \rtimes \langle t\rangle$ (see [Gor68, Theorem 9.1.1]). Seeing $\widetilde{H}$ as generated by yt and t, it follows immediately that $\widetilde{H}$ has a dihedral subgroup of order 6, say H generated by t and a suitable power of yt. The claim follows. $\square$

Corollary 4.3. *Let G be a finite group with a dihedral 2-subgroup P of order at least 8, and let $Q \lneq P$ such that $Q \cong C_2 \times C_2$. Assume moreover that $C_G(Q)$ is 2-nilpotent and that $N_G(Q)/C_G(Q) \cong \mathfrak{S}_3$. Then there exists a subgroup H of $N_G(Q)$ such that $N_P(Q)$ is a Sylow 2-subgroup of H and $|N_G(Q) : H|$ is a power of 2 (possibly 1).*

Proof. Let $K := O_{2'}(C_G(Q))$ and let $R \in \mathrm{Syl}_2(C_G(Q))$. Then by assumption we have the following inclusions of subgroups

$$\begin{tikzcd}
 & N_G(Q) \arrow[dl, dash, "\mathfrak{S}_3"'] \arrow[ddr, dash] & \\
C_G(Q) = K \rtimes R \arrow[ddr, dash] & & \\
 & & K \rtimes N_P(Q) = K \rtimes D_8 \arrow[dl, dash, "C_2"] \\
 & K \times Q = K \rtimes (C_2 \times C_2) &
\end{tikzcd}$$

where we note that $|N_P(Q)/Q| = 2$ by [Bra74, Proposition (1B)], hence $K \rtimes N_P(Q) = K \rtimes D_8$. Since $[K, Q] = 1$ by the choice of K, we have $K \rtimes Q = K \times Q$. Furthermore, as K is a characteristic subgroup of $C_G(Q)$ and $C_G(Q) \lhd N_G(Q)$, we have $K \lhd N_G(Q)$, so that $(K \times Q) \lhd N_G(Q)$. Clearly $(K \times Q) \lhd (K \rtimes N_P(Q))$ since the index is two. Also $C_G(Q) \cap (K \rtimes N_P(Q)) = K \times Q$. Therefore we can take quotients by $L := K \times Q$ of all the groups in the picture. This yields the following inclusions of subgroups:

$$\begin{tikzcd}
 & \mathsf{G} := N_G(Q)/L \arrow[dl, dash, "\mathfrak{S}_3"'] \arrow[ddr, dash] & \\
\mathsf{Q} := C_G(Q)/L \arrow[ddr, dash] & & \\
 & & \mathsf{R} := (K \rtimes N_P(Q))/L \arrow[dl, dash, "C_2"] \\
 & \langle 1\rangle = (K \times Q)/L = \mathsf{Q} \cap \mathsf{R} &
\end{tikzcd}$$

In particular, we have that $\mathsf{G}/\mathsf{Q} \cong \mathfrak{S}_3$, $\mathsf{R} = (K \rtimes N_P(Q))/L \cong N_P(Q)/Q \cong C_2$, $\mathsf{Q} = (K \rtimes R)/(K \times Q) \cong R/Q$ (which is a 2-group), and $\mathsf{Q} \cap \mathsf{R} = \langle 1\rangle$. Now there must exist an involution $\mathsf{t} \in \mathsf{R}$ such that $\mathsf{t} \notin \mathsf{Q}$. Therefore, by Lemma 4.2, there exists a subgroup H of

G such that $\mathsf{t} \in \mathsf{H} \cong \mathfrak{S}_3$. Finally, we set H to be the preimage of H under the canonical homomorphism $N_G(Q) \twoheadrightarrow N_G(Q)/L$ and the claim follows. $\square$

We can now prove Theorem 1.3 of the Introduction:

Proof of Theorem 1.3. Set $M := \mathrm{Sc}(G, P)$. Let $Q \leq P$ be an arbitrary fully normalised subgroup in $\mathcal{F}_P(G)$. We claim that if $Q \neq 1$, then $N_G(Q)$ has a subgroup H_Q which satisfies the conditions (1) and (2) in Theorem 2.4.

First suppose that $Q \ncong C_2 \times C_2$. Then, $\mathrm{Aut}(Q)$ is a 2-group (see [Gor68, Lemma 5.4.1 (i)-(ii)]), and hence $N_G(Q)/C_G(Q)$ is also a 2-group. Thus $N_G(Q)$ is 2-nilpotent since $C_G(Q)$ is 2-nilpotent by the assumption. Set $N := N_G(Q)$, and hence we can write $N := K \rtimes P_N$ where $K := O_{2'}(N)$ and P_N is a Sylow 2-subgroup of N. Since $N_P(Q)$ is a 2-subgroup of N, we can assume $P_N \geq N_P(Q)$. Set $H_Q := K \rtimes N_P(Q)$. Then, obviously $N_P(Q)$ is a Sylow 2-subgroup of H_Q and $|N : H_Q|$ is a power of 2 (possibly one) since $|N : H_Q| = |P_N : N_P(Q)|$. This means that Q satisfies the conditions (1) and (2) in Theorem 2.4.

Next suppose that $Q \cong C_2 \times C_2$. Then, $N_G(Q)/C_G(Q) \hookrightarrow \mathrm{Aut}(Q) \cong \mathfrak{S}_3$. Clearly $C_P(Q) = Q$ and $N_P(Q) \cong D_8$ (see [Bra74, Proposition (1B)]). Then, as $N_P(Q)/C_P(Q) \hookrightarrow N_G(Q)/C_G(Q)$, we have that $|N_G(Q)/C_G(Q)| \in \{2, 6\}$. If $|N_G(Q)/C_G(Q)| = 2$, then $N_G(Q)$ is 2-nilpotent since $C_G(Q)$ is 2-nilpotent, so that using an argument similar to the one in the previous case there exists a subgroup H_Q of $N_G(Q)$ such that H_Q satisfies the conditions (1) and (2) in Theorem 2.4. Hence we can assume that $|N_G(Q)/C_G(Q)| \neq 2$. Now assume that $|N_G(Q)/C_G(Q)| = 6$, so that $N_G(Q)/C_G(Q) \cong \mathfrak{S}_3$. Then it follows from Corollary 4.3 that $N_G(Q)$ has a subgroup H_Q such that $N_P(Q)$ is a Sylow 2-subgroup of H_Q and that $|N_G(Q) : H_Q|$ is a power of 2, as required. Therefore, by Theorem 2.4, $\mathrm{Sc}(N_G(Q), N_P(Q)) \downarrow^{N_G(Q)}_{QC_G(Q)}$ is indecomposable for each fully normalised subgroup $1 \neq Q \leq P$.

Now if $Q = 1$, then $\mathrm{Sc}(N_G(Q), N_P(Q)) \downarrow^{N_G(Q)}_{QC_G(Q)} = \mathrm{Sc}(G, P)$, which is indecomposable by definition.

Therefore Theorem 2.3 yields that M is Brauer indecomposable. $\square$

Corollary 4.4. *Let G and G' be finite groups with a common Sylow 2-subgroup P which is a dihedral group of order at least 8 and assume that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Then the Scott module $\mathrm{Sc}(G \times G', \Delta P)$ is Brauer indecomposable.*

Proof. Set $M := \mathrm{Sc}(G \times G', \Delta P)$. Since P is a Sylow 2-subgroup of G, $\mathcal{F}_P(G)$ is saturated by [BLO03, Proposition 1.3]. Therefore, as $\mathcal{F}_{\Delta P}(G \times G') \cong \mathcal{F}_P(G)$ by definition, we have that $\mathcal{F}_{\Delta P}(G \times G')$ is saturated.

Now let $\mathcal{Q} \leq \Delta P$ be any fully normalised subgroup in $\mathcal{F}_{\Delta P}(G \times G')$. Obviously we can write $\mathcal{Q} =: \Delta Q$ for a subgroup $Q \leq P$. We claim that the Brauer construction $M(\Delta Q)$ is indecomposable as a $k[C_{G \times G'}(\Delta Q)]$-module. Notice that clearly $C_{G \times G'}(\Delta Q) = C_G(Q) \times C_{G'}(Q)$.

If $Q = 1$, then the claim is obvious since $M(\Delta\langle 1 \rangle) = M$. So, assume that $Q \neq 1$. Hence Q contains an involution t. Thus, $C_G(Q) \leq C_G(t)$, so that [Bra66, Lemma (7A)] implies that $C_G(Q)$ is 2-nilpotent, and similarly for $C_{G'}(Q)$. Hence $C_{G \times G'}(\Delta Q)$ is 2-nilpotent. Therefore it follows from Theorem 1.3 that M is Brauer indecomposable. $\square$

Lemma 4.5. *Assume that G and G' are finite groups with a common Sylow 2-subgroup P which is a dihedral group of order at least 8. Assume, moreover, that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$ and is such that there are exactly two G-conjugacy classes of involutions in P. Further* *suppose that $z \in Z(P)$ and $t \in P$ are two involutions in P which are not G-conjugate. Set $M := \mathrm{Sc}(G \times G', \Delta P)$. Then*

$$M(\Delta\langle t\rangle) \cong \mathrm{Sc}(C_G(t) \times C_{G'}(t), \Delta C_P(t))$$

as $k[C_G(t) \times C_{G'}(t)]$-modules.

Proof. Set $\mathcal{F} := \mathcal{F}_{\Delta P}(G \times G')$. By the beginning of the proof of Lemma 3.2, $\mathcal{F}$ is saturated. Set $Q := \langle t\rangle$.

Next we claim that Q is a fully $\mathcal{F}_P(G)$-normalised subgroup of P. Let $R \leq P$ be $\mathcal{F}_P(G)$-conjugate to Q. So that R and Q are G-conjugate. So, we can write $R := \langle r\rangle$ for an element $r \in R$ since $|Q| = 2$. Thus, t and r are G-conjugate, which implies that r and z are not G-conjutate by the assumption, and hence r and t are P-conjugate again by the assumption. Namely, $t = r^\pi$ for an element $\pi \in P$. Obviously, $C_P(r)^\pi = C_{P^\pi}(r^\pi) = C_P(r^\pi) = C_P(t)$, so that $|C_P(r)| = |C_P(t)|$, which yields that $|N_P(R)| = |N_P(Q)|$ since $|R| = |Q| = 2$. Therefore Q is a fully $\mathcal{F}_P(G)$-noramlised subgroup of P.

Thus, ΔQ is a fully $\mathcal{F}$-normalised subgroup of ΔP since $\mathcal{F} \cong \mathcal{F}_P(G) = \mathcal{F}_P(G')$ (see the beginning of the proof of Lemma 3.2). Moreover, M is Brauer indecomposable by Lemma 4.4. Hence it follows from [IK17, Theorem 3.1] that

$$M(\Delta Q) \cong \mathrm{Sc}(N_{G\times G'}(\Delta Q), N_{\Delta P}(\Delta Q))$$

as $k[N_{G\times G'}(\Delta Q)]$-modules. Noting that $|\Delta Q| = 2$ and $\Delta Q = \Delta\langle t\rangle$, we have that

$$N_{G\times G'}(\Delta Q) = C_{G\times G'}(\Delta Q) = C_G(Q) \times C_{G'}(Q) = C_G(t) \times C_{G'}(t)$$

and that

$$N_{\Delta P}(\Delta Q) = C_{\Delta P}(\Delta Q) = \Delta C_P(Q) = \Delta C_P(t).$$

The assertion follows. □

Proposition 4.6. *Let G and G' be finite groups with a common Sylow 2-subgroup P which is a dihedral group of order at least 8. Assume moreover that $O_{2'}(G) = O_{2'}(G') = 1$ and $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Then the Scott module $\mathrm{Sc}(G\times G', \Delta P)$ induces a stable equivalence of Morita type between the principal blocks $B_0(kG)$ and $B_0(kG')$.*

Proof. Fix $P = D_{2^n}$ for an $n \geq 3$, and set $M := \mathrm{Sc}(G \times G', \Delta P)$, $B := B_0(kG)$ and $B' := B_0(kG')$. Since we assume that $O_{2'}(G) = O_{2'}(G') = 1$, G and G' are amongst the groups (D1)-(D3) listed in §2.4. First we note that $G = P$ is the unique group in this list with $\mathcal{F}_P(G) = \mathcal{F}_P(P)$, therefore we may assume that $G \neq P \neq G'$. Thus, by §2.5 and by the assumption that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$, we have that P has the same number of G-conjugacy and G'-conjugacy classes of involutions, namely either one or two.

Let $t \in P$ be an arbitrary involution, and set $B_t := B_0(kC_G(t))$ and $B'_t := B_0(kC_{G'}(t))$. We claim that $M(\Delta\langle t\rangle)$ induces a Morita equivalence between B_t and B'_t.

First of all, assume that $t \in Z(P)$. Set $z := t$. Thus $C_P(z) = P$ and this is a Sylow 2-subgroup of $C_G(z)$, so that P is a Sylow 2-subgroup of $C_{G'}(z)$ as well. Recall that $C_G(t)$ and $C_{G'}(t)$ are both 2-nilpotent by [Bra66, Lemma (7A)]. Set

$$M_z := \mathrm{Sc}(C_G(z) \times C_{G'}(z), \Delta P).$$

By Lemma 3.1, M_z induces a Morita equivalence between B_z and B'_z. Hence we have $M_z \mid M(\Delta\langle z\rangle)$ by Lemma 3.2 and Corollary 4.4 yields

$$M(\Delta\langle z\rangle) = M_z.$$

This means that $M(\Delta\langle z\rangle)$ induces a Morita equivalence between B_z and B'_z.

Case 1: Assume first that all involutions in P are G-conjugate, so that all involutions in P are G'-conjugate a well, since $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. Therefore there exists an element $g \in G$ and an element $g' \in G'$ such that $z^g = t = z^{g'}$. Thus, by definition of the Brauer construction, we have

$$M(\Delta\langle t\rangle) = M(\Delta\langle z^g\rangle) = M((\Delta\langle z\rangle)^{(g,g')}) = M(\Delta\langle z\rangle)^{(g,g')} = (M_z)^{(g,g')}.$$

Moreover, we have

$$\begin{aligned}(M_z)^{(g,g')} &= (\mathrm{Sc}(C_G(z) \times C_{G'}(z),\, \Delta P))^{(g,g')}\\ &= \mathrm{Sc}(C_G(z^g) \times C_{G'}(z^{g'}),\, (\Delta P)^{(g,g')})\\ &= \mathrm{Sc}(C_G(t) \times C_{G'}(t),\, \Delta)\,,\end{aligned}$$

where $\Delta := \{(\pi^g, \pi^{g'}) \mid \pi \in P\} \cong P$. Obviously P^g is a Sylow 2-subgroup of $C_G(t)$ and $P^{g'}$ is a Sylow 2-subgroup of $C_{G'}(t)$. Further $C_G(t)$ and $C_{G'}(t)$ are 2-nilpotent. Therefore it follows from Lemma 3.1 that

$$\mathrm{Sc}(C_G(t) \times C_{G'}(t),\, \Delta) = M(\Delta\langle t\rangle)$$

induces a Morita equivalence between B_t and B'_t.

Case 2: Assume now that P has exactly two G-conjugacy classes and hence exactly two G'-conjugacy classes of involutions. Then, by §2.5, G and G' are groups of type (D3)(ii), that is $G \cong \mathrm{PGL}_2(q) \rtimes C_f$ and $G' \cong \mathrm{PGL}_2(q') \rtimes C_{f'}$ for some odd prime powers q, q' and some suitable odd positive integers f, f'.

If t is G-conjugate to the central element $z \in Z(P)$, then $M(\Delta\langle t\rangle)$ induces a Morita equivalence between B_t and B'_t by the same argument as in Case 1. Hence we may assume that t is not G-conjugate to z. We note that by Lemma 2.5(a) any two involutions in P which are not G-conjugate (resp. G'-conjugate) to z are already P-conjugate. It follows easily from Lemma 2.5(b) that $C_P(t)$ is a Sylow 2-subgroup of both $C_G(t)$ and $C_G(t')$. Again, because $C_G(t)$ and $C_{G'}(t)$ are both 2-nilpotent, it follows from Lemma 3.1 that

$$M_t := \mathrm{Sc}(C_G(t) \times C_{G'}(t),\, \Delta C_P(t))$$

induces a stable equivalence of Morita type between B and B'.

On the other hand, Lemma 4.5 implies that

$$M(\Delta\langle t\rangle) = M_t\,.$$

Therefore $M(\Delta\langle t\rangle)$ induces a Morita equivalence between B_t and B'_t. Hence the claim holds.

Finally Lemma 4.1 yields that M induces a stable equivalence of Morita type between B and B'. □

Corollary 4.7. *Let G and G' be two finite groups with a common Sylow 2-subgroup $P \cong D_{2^n}$ with $n \geq 3$ and let $M := \mathrm{Sc}(G \times G', \Delta P)$.*

(a) *If $G = \mathrm{PSL}_2(q)$ and $G' = \mathrm{PSL}_2(q')$, where q and q' are powers of odd primes such that $q \equiv q' \pmod 8$ and $|G|_2 = |G'|_2 \geq 8$, then M induces a stable equivalence of Morita type between $B_0(kG)$ and $B_0(kG')$.*

(b) *If $n = 3$, $G = \mathfrak{A}_7$ and $G' = \mathrm{PSL}_2(q')$, where q' is a power of an odd prime such that $|G'|_2 = 8$, then M induces a stable equivalence of Morita type between $B_0(kG)$ and $B_0(kG')$.*

(c) *If* $G = \mathrm{PGL}_2(q)$ *and* $G' = \mathrm{PGL}_2(q')$, *where* q *and* q' *are powers of odd primes such that* $q \equiv q' \pmod 4$ *and* $|G|_2 = |G'|_2$, *then* M *induces a stable equivalence of Morita type between* $B_0(kG)$ *and* $B_0(kG')$.

Furthermore, there exists a stable equivalence of Morita type between $B_0(kG)$ *and* $B_0(kG')$ *if and only if* $\mathcal{F}_P(G) = \mathcal{F}_P(G')$.

Proof. Parts (a), (b) and (c) follow directly from Proposition 4.6 since in each case $\mathcal{F}_P(G) = \mathcal{F}_P(G')$. The sufficient condition of the last statement also follows from Proposition 4.6 since we have already noticed that inflation induces splendid Morita equivalences (hence stable equivalences of Morita type) between $B_0(kG)$ and $B_0(kG/O_{2'}(G))$, resp. between $B_0(kG')$ and $B_0(kG/O_{2'}(G'))$. To prove the necessary condition, we recall that the existence of a stable equivalence of Morita type between $B_0(kG)$ and $B_0(kG')$ implies that

$$k(B_0(G)) - l(B_0(G)) = k(B_0(G')) - l(B_0(G'))$$

(see [Bro94, 5.3.Proposition]). Using [Bra66, Theorem (7B)], we have that $k(B_0(G)) = k(B_0(G')) = 2^{n-2} + 3$, so that we must have $l(B_0(G)) = l(B_0(G'))$, which in turn forces $\mathcal{F}_P(G) = \mathcal{F}_P(G')$; see §2.5. □

5. The principal blocks of $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$

Throughout this section we assume that k is a field of characteristic 2. We now start to determine when the stable equivalences of Morita type we constructed in the previous section are actually Morita equivalences, and in consequence splendid Morita equivalences. We note that these Morita equivalences are known from the work of Erdmann [Erd90] (over k) or Plesken [Ple83, VII] (over complete discrete valuation rings), but the methods used do not prove that they are *splendid* Morita equivalences.

We start with the case $\mathrm{PSL}_2(q)$ and we fix the following notation: we set $\mathbb{B}(q) := C/Z(\mathrm{SL}_2(q))$, where $C \leq \mathrm{SL}_2(q)$ is the subgroup of upper-triangular matrices. We have $|\mathrm{PSL}_2(q)| = \frac{1}{2}q(q-1)(q+1)$ and $|\mathbb{B}(q)| = \frac{1}{2}q(q-1)$. Furthermore, the principal 2-block of $\mathrm{PSL}_2(q)$ contains three simple modules, namely the trivial module and two mutually dual modules of k-dimension $(q-1)/2$, which we denote by $S(q)$ and $S(q)^*$. (See e.g. [Erd77, Lemma 4.3 and Corollary 5.2]). Throughout, we heavily rely heavily on Erdmann's computation of the PIMs of $\mathrm{PSL}_2(q)$ in [Erd77].

Lemma 5.1. *Let* $G := \mathrm{PSL}_2(q)$, *where* q *is a power of an odd prime, and let* P *be a Sylow* 2*-subgroup of* G. *Then the Loewy and socle series structure of the Scott module with respect to* $\mathbb{B}(q)$ *is*

$$\mathrm{Sc}(G, \mathbb{B}(q)) = \boxed{\begin{matrix} k_G \\ S(q) \oplus S(q)^* \\ k_G \end{matrix}}\,.$$

Before proceeding with the proof, we note that in this lemma we allow the Sylow 2-subgroups to be Klein-four groups as this case will be necessary when dealing with the groups of type $\mathrm{PGL}_2(q)$ and dihedral Sylow 2-subgroups of order 8.

Proof. Assume first that $q \equiv -1 \pmod 4$. Then $2 \nmid |\mathbb{B}(q)|$, so that $\mathrm{Sc}(G, \mathbb{B}(q)) = P(k_G)$ (see [NT88, Corollary 4.8.5]). Therefore, by [Erd77, Theorem 4(a)], the Loewy and socle structure of the PIM $P(k_G) = \mathrm{Sc}(G, \mathbb{B}(q))$ is as claimed.

Assume next that $q \equiv 1 \pmod 4$. By [Bon11, §3.2.3] the trivial source module $k_{\mathbb{B}(q)}{\uparrow}^G$ affords the ordinary character

$$1_{\mathbb{B}(q)}{\uparrow}^G = 1_G + \mathrm{St}_G\,, \tag{5}$$

where St_G denotes the Steinberg character. Therefore $k_{\mathbb{B}(q)}{\uparrow}^G$ is indecomposable and isomorphic to $\mathrm{Sc}(G, \mathbb{B}(q)) =: X$. Then it follows from [Bon11, Table 9.1] that

$$X = k_G + (k_G + S(q) + S(q)^*)$$

as composition factors. Since X is an indecomposable self-dual 2-permutation kG-module, its Loewy and socle structure is one of:

$$\boxed{\begin{matrix} k_G \\ S(q) \oplus S(q)^* \\ k_G \end{matrix}}\,, \qquad \boxed{\begin{matrix} k_G \\ S(q) \\ S(q)^* \\ k_G \end{matrix}}\,, \qquad \boxed{\begin{matrix} k_G \\ S(q)^* \\ S(q) \\ k_G \end{matrix}}\,, \qquad \boxed{\begin{matrix} k_G \oplus S(q) \\ k_G \oplus S(q)^* \end{matrix}} \qquad \text{or} \qquad \boxed{\begin{matrix} k_G \oplus S(q)^* \\ k_G \oplus S(q) \end{matrix}}\,.$$

By [Erd77, Theorem 2(a)], $\mathrm{Ext}^1_{kG}(S(q), S(q)^*) = 0 = \mathrm{Ext}^1_{kG}(S(q)^*, S(q))$, hence the second and the third cases cannot occur.

Suppose now that the fourth case happens. Then X has a submodule Y such that $X/Y \cong k_G$. Hence $Y = S(q) + S(q)^* + k_G$ as composition factors. Then, since $\mathrm{Ext}^1_{kG}(S(q), S(q)^*) = 0$, Y has the following structure:

$$Y = \boxed{\begin{matrix} S(q) \\ k_G \end{matrix}} \oplus S(q)^*$$

Hence Y has a submodule Z with the Loewy and socle structure

$$Z = \boxed{\begin{matrix} S(q) \\ k_G \end{matrix}}\,.$$

Similarly X has a submodule W such that $X/W \cong S(q)$, and $W = k_G + k_G + S(q)^*$ as composition factors. By [Erd77, Theorem 2], $\mathrm{Ext}^1_{kG}(k_G, k_G) = 0$. Therefore W has the following structure:

$$W = \boxed{\begin{matrix} k_G \\ S(q)^* \end{matrix}} \oplus k_G\,,$$

and hence W has a submodule U with structure

$$U = \boxed{\begin{matrix} k_G \\ S(q)^* \end{matrix}}\,.$$

Since Z and U are submodules of X and $Z \cap U = 0$, we have a direct sum $Z \oplus U$ in X. As a consequence $X = Z \oplus U$, which is a contradiction since X is indecomposable. Hence the fourth case cannot occur.

Similarly, the fifth case cannot happen. Therefore we must have that

$$\mathrm{Sc}(G, \mathbb{B}(q)) = \boxed{\begin{matrix} k_G \\ S(q) \oplus S(q)^* \\ k_G \end{matrix}}$$

as desired. □

Proposition 5.2. *Let $G := \mathrm{PSL}_2(q)$ and $G' := \mathrm{PSL}_2(q')$, where q and q' are powers of odd primes such that $q \equiv q' \pmod 4$ and $|G|_2 = |G'|_2 \geq 8$. Let P be a common Sylow 2-subgroup of G and G'. Then the Scott module $\mathrm{Sc}(G\times G',\, \Delta P)$ induces a splendid Morita equivalence between $B_0(kG)$and $B_0(kG')$.*

Proof. Set $M := \mathrm{Sc}(G \times G', \Delta P)$ and $B := B_0(kG)$ and $B' := B_0(kG')$. First, by Proposition 4.7(a), M induces a stable equivalence of Morita type between B and B'. We claim that this is a Morita equivalence. Using Theorem 2.1(b), it is enough to check that the simple B-modules are mapped to the simple B'-modules.

To start with, by Proposition 4.7(a) and Lemma 3.4(a), we have

$$k_G \otimes_B M \;=\; k_{G'}\,. \tag{6}$$

Next, because $q \equiv q' \pmod 4$, the Scott modules $\mathrm{Sc}(G,\, \mathbb{B}(q))$ and $\mathrm{Sc}(G',\, \mathbb{B}(q'))$ have a common vertex Q (which depend on the value of q modulo 4). Therefore, by Lemma 3.4(c), we have

$$\mathrm{Sc}(G,\, \mathbb{B}(q)) \otimes_B M = \mathrm{Sc}(G',\, \mathbb{B}(q')) \oplus (\mathrm{proj})\,.$$

Moreover, as $\mathrm{Sc}(G,\, \mathbb{B}(q))$ and $\mathrm{Sc}(G',\, \mathbb{B}(q'))$ are the relative Q-projective covers of k_G and $k_{G'}$ respectively, it follows from Lemma 5.1 that the socle series of $\Omega_Q(k_G)$ and $\Omega_Q(k_{G'})$ are given by

$$\Omega_Q(k_G) \;=\; \boxed{\begin{matrix} S(q)\oplus S(q)^* \\ k_G \end{matrix}} \quad \text{and} \quad \Omega_Q(k_{G'}) \;=\; \boxed{\begin{matrix} S(q')\oplus S(q')^* \\ k_{G'} \end{matrix}}\,.$$

Thus, by Lemma 3.4(d),

$$\boxed{\begin{matrix} S(q)\oplus S(q)^* \\ k_G \end{matrix}} \otimes_B M = \boxed{\begin{matrix} S(q')\oplus S(q')^* \\ k_{G'} \end{matrix}} \oplus\ (\mathrm{proj})\ .$$

Then it follows from (6) and [KMN11, Lemma A.1] (stripping-off method) that

$$(S(q) \otimes_B M) \oplus (S(q)^* \otimes_B M) = (S(q) \oplus S(q)^*) \otimes_B M = S(q') \oplus S(q')^* \oplus (\mathrm{proj}).$$

Thus Theorem 2.1(a) implies that both (non-projective) simple B-modules $S(q)$ and $S(q)^*$ are mapped to a simple B'-module.

In conclusion, Theorem 2.1(b) yields that M induces a Morita equivalence. As M is a 2-permutation $k(G \times G')$-module, the Morita equivalence induced by M is actually a splendid Morita equivalence, see §2.2. □

Next we consider the case $\mathrm{PGL}_2(q)$. We fix a subgroup $H(q) < \mathrm{PGL}_2(q)$ such that $H(q) \cong \mathrm{PSL}_2(q)$ and keep the notation $\mathbb{B}(q) < H(q)$ as above. Furthermore, the principal 2-block of $\mathrm{PGL}_2(q)$ contains two simple modules, namely the trivial module and a self-dual module of dimension $q-1$, which we denote by $T(q)$.

Lemma 5.3. *Let $G := \mathrm{PGL}_2(q)$ where q is a power of an odd prime. Then the Loewy and socle sturucture of the Scott module $\mathrm{Sc}(G,\, \mathbb{B}(q))$ with respect to $\mathbb{B}(q)$ is as follows:*

$$\mathrm{Sc}(G,\, \mathbb{B}(q)) \;=\; \boxed{\begin{matrix} k_G \\ \boxed{\begin{matrix} k_G \oplus T(q) \\ T(q) \oplus k_G \end{matrix}} \\ k_G \end{matrix}}$$

Proof. Write $H := H(q)$, $\mathbb{B} := \mathbb{B}(q)$, $Y := \mathrm{Sc}(H, \mathbb{B})$ and $X := Y{\uparrow}^G$. Since $|G/H| = 2$, Green's indecomposability theorem and Frobenius Reciprocity imply that $X = \mathrm{Sc}(G, \mathbb{B})$.

Let χ be the ordinary character of G afforded by the 2-permutation kG-module X. It follows from equation (5), [Bon11, Table 9.1] and Clifford theory, that

$$\chi \;=\; 1_{\mathbb{B}}{\uparrow}^H{\uparrow}^G \;=\; 1_G + 1' + \chi_{St_1} + \chi_{St_2} \tag{7}$$

where 1_G is the trivial character, $1'$ is the non-trivial linear character of G and χ_{St_i} for $i = 1, 2$ are the two distinct irreducible constituents of $\chi_{St}{\uparrow}^G$ of degree q. Using [Bon11, Table 9.1] we have that the 2-modular reduction of X is

$$X = k_G + k_G + (k_G + T(q)) + (k_G + T(q))$$

as composition factors. Moroeover, by Proposition 5.1, we have that the Loewy and socle series of Y is

$$Y = \boxed{\begin{matrix} k_H \\ S(q) \oplus S(q)^* \\ k_H \end{matrix}}\,.$$

Assume first that $q \equiv -1 \pmod 4$. As in the proof of Lemma 5.1, as $2 \nmid \mathbb{B}|$, we have that

$$P(k_G) = \mathrm{Sc}(G, \mathbb{B})\,.$$

Moreover, by Webb's theorem [Web82, Theorem E], the heart $\mathcal{H}(P(k_G))$ of $P(k_G)$ is decomposable with precisely two indecomposable summands and by [AC86, Lemma 5.4 and Theorem 5.5], these two summands are dual to each other and endo-trivial modules. It follows that $P(k_G)$ must have the following Loewy and socle structure:

$$\mathrm{Sc}(G, \mathbb{B}) = P(k_G) = \boxed{\begin{matrix} k_G \\ \boxed{\begin{matrix} k_G \\ T(q) \end{matrix}} \oplus \boxed{\begin{matrix} T(q) \\ k_G \end{matrix}} \\ k_G \end{matrix}}\,.$$

Next assume that $q \equiv 1 \pmod 4$. Since $X = Y{\uparrow}^G$, $T(q) = S(q){\uparrow}^G = S(q)^*{\uparrow}^G$, Frobenius Reciprocity implies that $X/(X\,\mathrm{rad}(kG)) \cong \mathrm{soc}(X) \cong k_G$, and that X has a filtration by submodules such that $X \gneqq X_1 \gneqq X_2$ such that

$$X/X_1 \cong \boxed{\begin{matrix} k_G \\ k_G \end{matrix}}, \quad X_1/X_2 \cong T(q) \oplus T(q) \quad \text{and} \quad X_2 \cong \boxed{\begin{matrix} k_G \\ k_G \end{matrix}}$$

by Proposition 5.1. Since X is a 2-permutation kG-module, we know by (7) and Scott's theorem on the lifting of homomorphisms (see [Lan83, Theorem II.12.4(iii)]) that

$$\dim[\mathrm{Hom}_{kG}(X, \boxed{\begin{matrix} k_G \\ k_G \end{matrix}})] \;= 2.$$

This implies that X has both a factor module and a submodule which have Loewy structure:

$$\boxed{\begin{matrix} k_G \\ k_G \end{matrix}}\,.$$

Now we note that k_G occurs exactly once in the second Loewy layer of X as $|G/O^2(G)| = 2$, and we have that the Loewy structure of X is of the form

$$X = \boxed{\begin{array}{c} k_G \\ k_G \ \cdots \\ \vdots \\ k_G \end{array}}$$

so that only $k_G + (2 \times T(q))$ are left to determine. Further we know from Shapiro's Lemma and [Erd77, Theorem 2] that $\mathrm{Ext}^1_{kG}(T(q), T(q)) = 0$ and that

$$\dim[\mathrm{Ext}^1_{kG}(k_G, T(q))] = \dim[\mathrm{Ext}^1_{kG}(T(q), k_G)] = 1\,.$$

Hence, using the above filtration of X, we obtain that the Loewy structure of X/X_2 is

$$X/X_2 = \boxed{\begin{array}{c} k_G \\ k_G \oplus T(q) \\ T(q) \end{array}}\,. \tag{8}$$

Thus X has Loewy structure

$$\boxed{\begin{array}{c} k_G \\ k_G \oplus T(q) \\ T(q) \oplus k_G \\ k_G \end{array}} \quad \text{or} \quad \boxed{\begin{array}{c} k_G \\ k_G \oplus T(q) \\ T(q) \\ k_G \\ k_G \end{array}}\,. \tag{9}$$

It follows from (8) and the self-dualities of k_G, $T(q)$ and X that X has a submodule X_3 such that the socle series has the form

$$X_3 = \boxed{\begin{array}{c} T(q) \\ k_G \oplus T(q) \\ k_G \end{array}} \quad \text{(socle series)}. \tag{10}$$

First, assume that the second case in (9) holds. Then, again by the self-dualities, the socle series of X has the form

$$X = \boxed{\begin{array}{c} k_G \\ k_G \\ T(q) \\ k_G \oplus T(q) \\ k_G \end{array}} \quad \text{(socle series)}. \tag{11}$$

By making use of (9) and (11), we have
(12)

$$X = \boxed{\begin{array}{c} k_G \\ \boxed{\begin{array}{c} k_G \\ T(q) \\ k_G \end{array}} \oplus \boxed{T(q)} \\ k_G \end{array}} = \boxed{\begin{array}{c} k_G \\ k_G \oplus T(q) \\ T(q) \\ k_G \\ k_G \end{array}} \text{(Loewy series)} = \boxed{\begin{array}{c} k_G \\ k_G \\ T(q) \\ k_G \oplus T(q) \\ k_G \end{array}} \text{(socle series)}.$$

It follows from (12) that (up to isomorphism) there are exactly four factor modules $U_1, \ldots, U_4$ of X such that $U_i/(U_i\,\mathrm{rad}(kG)) \cong \mathrm{soc}(U_i) \cong k_G$ for each i, and furthermore that these have structures such that

$$(13) \qquad U_1 = X, \quad U_2 \cong k_G, \quad U_3 \cong \boxed{\begin{array}{c} k_G \\ k_G \end{array}}, \quad U_4 \cong \boxed{\begin{array}{c} k_G \\ k_G \\ T(q) \\ k_G \end{array}}.$$

Now we know by (12) that X has the following socle series

$$(14) \qquad X = \boxed{\begin{array}{c} k_G \\ k_G \\ T(q) \\ k_G \oplus T(q) \\ k_G \end{array}} \quad \text{(socle series)}.$$

If X has a submodule isomorphic to U_4, then this contradicts (14) by comparing the third (from the bottom) socle layers of U_4 and X, since $\mathrm{soc}^3(U_4) \cong k_G$ and $\mathrm{soc}^3(X) \cong T(q)$ (since U_4 is a submodule of X, $\mathrm{soc}^3(U_4) \hookrightarrow \mathrm{soc}^3(X)$ by [Lan83, Chap.I Lemma 8.5(i)]). This yields that such a U_4 does not exist as a submodule of X. Hence, by (13),

$$\dim_k[\mathrm{End}_{kG}(X)] = 3.$$

However, by (7) and Scott's theorem on lifting of endomorphisms of p-permutation modules [Lan83, Theorem II 12.4(iii)], this dimension has to be 4, so that we have a contradiction. As a consequence the second case in (9) does not occur. This implies that only the first case in (9) can occur. The claim follows. $\square$

Proposition 5.4. *Let $G := \mathrm{PGL}_2(q)$ and $G' := \mathrm{PGL}_2(q')$, where q and q' are powers of odd primes such that $q \equiv q' \pmod 4$ and $|G|_2 = |G'|_2$. Let P be a common Sylow 2-subgroup of G and G'. Then the Scott module $\mathrm{Sc}(G \times G', \Delta P)$ induces a splendid Morita equivalence between $B_0(kG)$ and $B_0(kG')$.*

Proof. Set $B := B_0(kG)$, $B' := B_0(kG')$, and $M := \mathrm{Sc}(G \times G', \Delta P)$. By Proposition 4.7(c), M induces a stable equivalence of Morita type between B and B'. Again we claim that this stable equivalence is a Morita equivalence. Let Q be a Sylow 2-subgroup of $\mathbb{B}(q)$. Then it follows from Lemma 5.3 that the Loewy structures of $\mathcal{H} := \mathcal{H}(P_Q(k_G)) := \Omega_Q(k_G)/k_G$ and $\mathcal{H}' := \mathcal{H}(P_Q(k_{G'})) := \Omega_Q(k_{G'})/k_{G'}$ are given by

$$\mathcal{H} = \boxed{\begin{array}{c} k_G \oplus T(q) \\ T(q) \oplus k_G \end{array}} \quad \text{and} \quad \mathcal{H}' = \boxed{\begin{array}{c} k_{G'} \oplus T(q') \\ T(q') \oplus k_{G'} \end{array}}.$$

Then it follows from Lemma 3.4(d) that

$$\boxed{\begin{array}{c} k_G \oplus T(q) \\ T(q) \oplus k_G \end{array}} \otimes_B M = \boxed{\begin{array}{c} k_{G'} \oplus T(q') \\ T(q') \oplus k_{G'} \end{array}} \oplus (\mathrm{proj}).$$

Thus by the stripping-off method [KMN11, Lemma A.1] and Lemma 3.4(a) we obtain that

$$(T(q) \oplus T(q)) \otimes_B M = T(q') \oplus T(q') \oplus (\mathrm{proj}).$$

Since $T(q)$ is non-projective, Theorem 2.1(a) implies that

$$(T(q) \oplus T(q)) \otimes_B M = T(q') \oplus T(q').$$

Hence $T(q) \otimes_B M = T(q')$. In addition, $k_G \otimes_B M = k_{G'}$ by Lemma 3.4(a). Finally, since $\mathrm{IBr}(B) = \{k_G, T(q)\}$ and $\mathrm{IBr}(B') = \{k_{G'}, T(q')\}$, it follows from Theorem 2.1(b) that M induces a Morita equivalence between B and B'. □

6. The principal blocks of $\mathrm{PSL}_2(q) \rtimes C_f$ and $\mathrm{PGL}_2(q) \rtimes C_f$.

Let $q := r^m$, where r is a fixed odd prime number and m is a positive integer. We now let H be one of the groups $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}_2(q)$, and we assume, moreover, that a Sylow 2-subgroup P of H is dihedral of order at least 8. We let $G := H \rtimes C_f$, where $C_f \leq \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_r)$ is as described in cases (D3)(i)-(ii) of Section 2.4. By the Frattini argument, we have $G = N_G(P)H$, therefore $G/H = N_G(P)H/H$ and we may assume that we have chosen notation such that the cyclic subgroup C_f normalises P.

Lemma 6.1. *With the notation above, we have $G = H\, C_G(P)$.*

Proof. The normaliser of P in G has the form

$$N_G(P) = P\, C_G(P) = P \times O_{2'}(C_G(P))$$

because $\mathrm{Aut}(P)$ is a 2-group and P a Sylow 2-subgroup of G (see e.g. [Gor68, Lemma 5.4.1 (i)-(ii)]). Therefore, by the above, the subgroup $C_f \leq G$ centralises P so that we must have

$$G = HC_f \leq H\, C_G(P)$$

and hence equality holds. □

We can now apply the result of Alperin and Dade (Theorem 2.2) in order to obtain splendid Morita equivalences.

Corollary 6.2. *The principal 2-blocks $B_0(kG)$ and $B_0(kH)$ are splendidly Morita equivalent.*

Proof. As $G = H\, C_G(P)$ by Lemma 6.1 the claim follows directly from Theorem 2.2. □

7. Proof of Theorem 1.1

Proof of Theorem 1.1. First because we consider principal blocks only, we may assume that $O_{2'}(G) = 1$. Therefore, we may assume that G is one of the groups listed in (D1)-(D3) in §2.4. Now it is known by the work of Erdmann [Erd90] that the principal blocks in (1)-(6) fall into distinct Morita equivalence classes. Therefore the claim follows directly from Corollary 6.2, and Propositions 5.2 and 5.4. □

8. Generalised 2-decomposition numbers

Brauer, in [Bra66, §VII], computes character values at 2-elements for principal blocks with dihedral defect groups up to signs $\delta_1, \delta_2, \delta_3$, thus providing us with the generalised decomposition matrices of such blocks up to the signs $\delta_1, \delta_2, \delta_3$. As a corollary to Theorem 1.1, we can now specify these signs. See also [Mur09, §6] for partial results in this direction.

Throughout this section, we assume that G is a finite group with a dihedral Sylow 2-subgroup $P := D_{2^n}$ of order $2^n \geq 8$, for which we use the presentation

$$P = \langle s, t \mid s^{2^{n-1}} = t^2 = 1, tst = s^{-1} \rangle .$$

Furthermore, we let ζ denote a primitive 2^{n-1}-th root of unity in $\mathbb{C}$, and we let $z := s^{2^{n-2}}$ (see §2.5).

For a 2-block B of G, we let $\mathcal{D}_{\mathrm{gen}}(B) \in \mathrm{Mat}_{k(B)\times k(B)}$ denote its generalised 2-decomposition matrix. In other words: let $S_2(G)$ denote a set of representatives of the G-conjugacy classes of the 2-elements in a fixed defect group of B. Let $u \in S_2(G)$, $H := C_G(u)$, and consider $\chi \in \mathrm{Irr}(G)$. Then the *generalised* 2-*decomposition numbers* are defined to be the uniquely determined algebraic integers $d^u_{\chi\varphi}$ such that

$$\chi(uv) = \sum_{\varphi\in\mathrm{IBr}(H)} d^u_{\chi\varphi}\varphi(v) \quad \text{for } v \in H_{2'}\,,$$

and we set $\mathcal{D}^u := (d^u_{\chi\varphi})_{\substack{\chi\in\mathrm{Irr}(B)\\ \varphi\in\mathrm{IBr}(b_u)}}$ where b_u is a 2-block of H such that $b_u^G = B$, so that

$$\mathcal{D}_{\mathrm{gen}}(B) = \left(d^u_{\chi\varphi} \mid \chi \in \mathrm{Irr}(B), u \in S_2(G), \varphi \in \mathrm{IBr}(b_u)\right)$$

is the *generalised* 2-*decomposition matrix* of B. We recall that $\mathcal{D}^1$ is simply the 2-decomposition matrix of B. By convention, we see $\mathcal{D}_{\mathrm{gen}}(B)$ as a matrix in $\mathrm{Mat}_{k(B)\times k(B)}$ via $\mathcal{D}_{\mathrm{gen}}(B) = (\mathcal{D}^u | u \in S_2(G))$ (one-row block matrix). Brauer [Bra66, Theorem (7B)] proved that $B_0(kG)$ satisfies

$$|\mathrm{Irr}(B_0(kG))| = 2^{n-2} + 3$$

and possesses exactly 4 characters of height zero ($\chi_0 := 1_G$, χ_1, χ_2, χ_3 in Brauer's notation [Bra66]). In the sequel we always label the 4 first rows of $\mathcal{D}_{\mathrm{gen}}(B_0)$ with these. The remaining characters are of height 1 and all have the same degree: unless otherwise specified, we denote them by $\chi^{(j)}$ with $1 \leq i \leq 2^{n-2}-1$, possibly indexed by their degrees. The first column of $\mathcal{D}_{\mathrm{gen}}(B_0(kG))$ is always labelled with the trivial Brauer character.

Corollary 8.1. *The principal* 2*-block* $B_0 := B_0(kG)$ *of a finite group* G *with a dihedral Sylow* 2*-subgroup* $P := D_{2^n}$ *of order* $2^n \geq 8$ *affords one of the following generalised decomposition matrices.*

(a) *If* $B_0 \sim_{SM} B_0(kD_{2^n})$, *then*

$$\mathcal{D}_{gen}(B_0) \ = \ \begin{array}{c|c:c:c:c:c} & \varphi_{1_1} & z = s^{2^{n-2}} & s^a & t & st \\ \hline 1_G & 1 & 1 & 1 & 1 & 1 \\ \chi_1 & 1 & 1 & 1 & -1 & -1 \\ \chi_2 & 1 & 1 & (-1)^a & 1 & -1 \\ \chi_3 & 1 & 1 & (-1)^a & -1 & 1 \\ \chi^{(j)} & 2 & 2(-1)^j & \zeta^{ja}+\zeta^{-ja} & 0 & 0 \end{array}$$

where $1 \leq j, a \leq 2^{n-2}-1$ *(up to relabelling the* χ_i*'s).*

(b) *If* $n = 3$ *and* $B_0 \sim_{SM} B_0(k\mathfrak{A}_7)$, *then*

$$\mathcal{D}_{gen}(B_0) \ = \ \begin{array}{c|ccc:c:c} & \varphi_{1_1} & \varphi_{14_1} & \varphi_{20_1} & z = s^2 & s \\ \hline 1_G & 1 & 0 & 0 & 1 & 1 \\ \chi_7 & 1 & 1 & 0 & -1 & -1 \\ \chi_8 & 1 & 0 & 1 & 1 & -1 \\ \chi_9 & 1 & 1 & 1 & -1 & 1 \\ \chi_5 & 0 & 1 & 0 & 2 & 0 \end{array}$$

where the irreducible characters and Brauer characters are labelled according to the ATLAS [CCN+85] *and the Modular Atlas* [WPT+11], *respectively.*

(c) *If* $B_0 \sim_{SM} B_0(k[\mathrm{PSL}_2(q)])$, *where* $(q-1)_2 = 2^n$, *then*

$$\mathcal{D}_{gen}(B_0) \;=\; \begin{array}{l|ccc:c:c}
 & \varphi_1 & \varphi_2 & \varphi_3 & z = s^{2^{n-2}} & s^a \\ \hline
1_G & 1 & 0 & 0 & 1 & 1 \\
\chi_1 = \chi^{(1)}_{(q+1)/2} & 1 & 1 & 0 & 1 & (-1)^a \\
\chi_2 = \chi^{(2)}_{(q+1)/2} & 1 & 0 & 1 & 1 & (-1)^a \\
\chi_3 = \chi_{St} & 1 & 1 & 1 & 1 & 1 \\
\chi^{(j)}_{q+1} & 2 & 1 & 1 & 2(-1)^j & \zeta^{ja} + \zeta^{-ja}
\end{array}$$

where $1 \leq j, a \leq 2^{n-2} - 1$, χ_1 *and* χ_2 *are labelled by their degrees, and* χ_{St} *is the Steinberg character.*

(d) *If* $B_0 \sim_{SM} B_0(k[\mathrm{PSL}_2(q)])$ *with* $(q+1)_2 = 2^n$, *then*

$$\mathcal{D}_{gen}(B_0) \;=\; \begin{array}{l|ccc:c:c}
 & \varphi_1 & \varphi_2 & \varphi_3 & z = s^{2^{n-2}} & s^a \\ \hline
1_G & 1 & 0 & 0 & 1 & 1 \\
\chi^{(1)}_{(q-1)/2} & 0 & 1 & 0 & -1 & (-1)^{a+1} \\
\chi^{(2)}_{(q-1)/2} & 0 & 0 & 1 & -1 & (-1)^{a+1} \\
\chi_{St} & 1 & 1 & 1 & -1 & -1 \\
\chi^{(j)}_{q-1} & 0 & 1 & 1 & 2(-1)^{j+1} & (-1)(\zeta^{ja} + \zeta^{-ja})
\end{array}$$

where $1 \leq j, a \leq 2^{n-2} - 1$, χ_1 *and* χ_2 *are labelled by their degrees, and* χ_{St} *is the Steinberg character.*

(e) *If* $B_0 \sim_{SM} B_0(k[\mathrm{PGL}_2(q)])$ *with* $2(q-1)_2 = 2^n$, *then*

$$\mathcal{D}_{gen}(B_0) \;=\; \begin{array}{l|cc:c:c:c}
 & \varphi_1 & \varphi_2 & t & z = s^{2^{n-2}} & s^a \\ \hline
1_G & 1 & 0 & 1 & 1 & 1 \\
\chi_1 = \chi^{(1)}_{q} & 1 & 1 & -1 & 1 & 1 \\
\chi_2 = \chi^{(2)}_{q} & 1 & 1 & 1 & 1 & (-1)^a \\
\chi_3 & 1 & 0 & -1 & 1 & (-1)^a \\
\chi^{(j)}_{q+1} & 2 & 1 & 0 & 2(-1)^j & \zeta^{ja} + \zeta^{-ja}
\end{array}$$

where $1 \leq j, a \leq 2^{n-2} - 1$, χ_1 *and* χ_2 *are labelled by their degrees, and* χ_3 *is a linear character.*

(f) *If* $B_0 \sim_{SM} B_0(k[\mathrm{PGL}_2(q)])$ *with* $2(q+1)_2 = 2^n$, *then*

$$\mathcal{D}_{gen}(B_0) \;=\; \begin{array}{l|cc:c:c:c}
 & \varphi_1 & \varphi_2 & t & z = s^{2^{n-2}} & s^a \\ \hline
1_G & 1 & 0 & 1 & 1 & 1 \\
\chi_1 = \chi^{(1)}_{q} & 1 & 1 & 1 & -1 & -1 \\
\chi_2 = \chi^{(2)}_{q} & 1 & 1 & -1 & -1 & (-1)^{a+1} \\
\chi_3 & 1 & 0 & -1 & 1 & (-1)^a \\
\chi^{(j)}_{q-1} & 0 & 1 & 0 & (-2)(-1)^j & (-1)(\zeta^{ja} + \zeta^{-ja})
\end{array}$$

where $1 \leq j, a \leq 2^{n-2} - 1$, χ_1 *and* χ_2 *are labelled by their degrees, and* χ_3 *is a linear character.*

Proof. Generalised decomposition numbers are determined by a source algebra of the block (see e.g. [Thé95, (43.10) Proposition]), hence they are preserved under splendid Morita equivalences. Thus, by Theorem 1.1, for a fixed defect group $P \cong D_{2^n}$ $(n \geq 3)$, if $n = 3$ there are exactly six, respectively, five if $n \geq 4$, generalised 2-decomposition matrices corresponding to cases (a) to (f) in Theorem 1.1.

Let $u \in S_2(G)$ be a 2-element. First if $u = 1_G$, then by definition $\mathcal{D}^u = \mathcal{D}^1$ is the 2-decomposition matrix of B_0. Therefore, in all cases, the necessary information about $\mathcal{D}^1$ is given either by Erdmann's work, see [Erd77, TABLES], or the Modular Atlas [WPT+11], or [Bon11, Table 9.1]. It remains to determine the matrices $\mathcal{D}^u$ for $u \neq 1$. As we consider principal blocks only, for each 2-element $u \in P$, the principal block b of $C_G(u)$ is the unique block of $C_G(u)$ with $b^G = B_0$ by Brauer's 3rd Main Theorem [NT88, Theorem 5.6.1]. Moreover, when $P = D_{2^n}$, then centralisers of non-trivial 2-elements always possess a normal 2-complement, so that their principal block is a nilpotent block [Bra64, Corollary 3]. It follows that $d^u_{\chi 1_H} = \chi(u)$.

These character values are given up to signs $\delta_1, \delta_2, \delta_3$ by [Bra66, Theorem (7B), Theorem (7C), Theorem (7I)]. Thus we can use the character tables of the groups D_{2^n}, $\mathfrak{A}_7$, $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$ (q odd), respectively, to determine the signs $\delta_1, \delta_2, \delta_3$.

(a) We may assume $G = D_{2^n}$. Since G is a 2-group, the generalised 2-decomposition matrix $\mathcal{D}_{\mathrm{gen}}(B_0)$ is the character table of G in this case. The claim follows.

(b) We may assume $G = \mathfrak{A}_7$. In this case $D = D_8$. Using the ATLAS [CCN+85], we have that the character table of B_0 at 2-elements is

	$1a$	$2a$	$4a$
1_G	1	1	1
χ_7	15	-1	-1
χ_8	21	1	-1
χ_9	35	-1	1
χ_5	14	2	0

.

Hence $\mathcal{D}_{\mathrm{gen}}(B_0)$ follows, using e.g. the Modular Atlas [WPT+11] to identify the rows of the matrix.

(c) We may assume $G = \mathrm{PSL}_2(q)$ with $(q-1)_2 = 2^n$. The height 0 characters in B_0 are 1_G, the Steinberg character χ_{St} (of degree q), and the two characters of G of degree $(q+1)/2$. Because the Steinberg character takes constant value 1 on s^r ($1 \leq a \leq 2^{n-2}$), we have $\delta_1 = 1$ in [Bra66, Theorem (7B)], so that $\chi^{(j)}(s^a) = \zeta^{ja} + \zeta^{-ja}$ for every $1 \leq a \leq 2^{n-2}$. And by [Bra66, Theorem (7C)], $\delta_2 = \delta_3 = -1$, from which it follows that $\chi^{(1)}_{(q+1)/2}(s^a) = \chi^{(1)}_{(q+1)/2}(s^a) = (-1)^a$ for each $1 \leq r \leq 2^{n-2}$. The claim follows.

(d) We may assume $G = \mathrm{PSL}_2(q)$ with $(q+1)_2 = 2^n$. The height 0 characters in B_0 are 1_G, the Steinberg character χ_{St} (of degree q), and the two characters of G of degree $(q-1)/2$. Because the Steinberg character takes constant value -1 on s^a ($1 \leq a \leq 2^{d-2}$), we have $\delta_1 = -1$ in [Bra66, Theorem (7B)], so that $\chi^{(j)}(s^a) = (-1)(\zeta^{ja}+\zeta^{-ja})$ for each $1 \leq a \leq 2^{d-2}$. Then by [Bra66, Theorem (7C)], $\delta_2 = \delta_3 = 1$, from which it follows that $\chi^{(1)}_{(q-1)/2}(s^a) = \chi^{(1)}_{(q-1)/2}(s^a) = (-1)^{a+1}$ for every $1 \leq a \leq 2^{d-2}$. The claim follows.

(e) We may assume $G = \mathrm{PGL}_2(q)$ with $2(q-1)_2 = 2^n$. The height one irreducible characters in B_0 have degree $q+1$. The four height zero irreducible characters in B_0 are the two linear characters 1_G and χ_3 (in Brauer's notation [Bra66, Theorem (7I)]) and the two characters of degree q, say $\chi^{(1)}_q$ and $\chi^{(2)}_q$.
Apart from 1_G, $\chi^{(1)}_q$ is the unique of these taking constant value 1 on s^a ($1 \leq a \leq 2^{d-2}$). Therefore using [Bra66, Theorem (7B), Theorem (7C), Theorem (7I)], we obtain $\delta_1 = 1$, $\delta_2 = -1 = \delta_3$. The claim follows.

(f) We may assume $G = \mathrm{PGL}_2(q)$ with $2(q+1)_2 = 2^n$. The height one irreducible characters in B_0 have degree $q-1$. The four height zero irreducible characters in B_0 are: the two linear characters 1_G and χ_3 (in Brauer's notation [Bra66, Theorem (7I)]) and the two characters of degree q, say $\chi_q^{(1)}$ and $\chi_q^{(2)}$.Apart from 1_G, $\chi_q^{(1)}$ is the unique of these taking constant value 1 on s^a $(1 \leq a \leq 2^{n-2})$. Therefore using [Bra66, Theorems (7B), (7C) and (7I)], we obtain $\delta_1 = -1$, $\delta_2 = 1$ and $\delta_3 = -1$. The claim follows.

For the character values of $\mathrm{PSL}_2(q)$, we refer to [Bon11] and for the character values of $\mathrm{PGL}_2(q)$, we refer to [Ste51]. □

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References

[AKO11] M. Aschebacher, R. Kessar, B. Oliver, *Fusion Systems in Algebra and Topology*, London Math. Soc. Lecture Note Series vol.**391**, 2011.

[AC86] M. Auslander, J. Carlson *Almost-split sequences and group rings*, J. Algebra **103** (1986), 122–140.

[Alp76] J.L. Alperin, *Isomorphic blocks*, J. Algebra **43** (1976), 694–698.

[Bon11] C. Bonnafé, *Representations of* $\mathrm{SL}_2(\mathbb{F}_q)$, Springer, Berlin, 2011.

[Bra64] R. Brauer, *Some applications of the theory of blocks of characters of finite groups I*, J. Algebra **1** (1964), 152–157.

[Bra66] R. Brauer, *Some applications of the theory of blocks of characters of finite groups III*, J. Algebra **3** (1966), 225–255.

[Bra74] R. Brauer, *On 2-blocks with dihedral defect groups*, in "Symposia Math.," Vol.13, pp.366–394, Academic Press, New York, 1974.

[BLO03] C. Broto, R. Levi, B. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), 779–856.

[Bro85] M. Broué, *On Scott modules and p-permutation modules*, Proc. Amer. Math. Soc. **93** (1985), 401–408.

[Bro94] M. Broué, *Equivalences of blocks of group algebras*, in "Finite-dimensional Algebras and Related Topics (Ottawa 1992)", Eds. V. Dlab and L.L. Scott, Kluwer Academic Pub., Dordrecht, 1994, pp.1–26.

[BP80] M. Broué, L. Puig, *Characters and local structure in G-algebras*, J. Algebra **63** (1980), 306–317.

[CCN+85] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[CEKL11] D.A. Craven, C.W. Eaton, R. Kessar, M. Linckelmann, *The structure of blocks with a Klein four defect group*, Math. Z. **268** (2011), 441–476.

[CG12] D.A. Craven, A. Glesser, *Fusion systems on small p-groups*, Trans. Amer. Math. Soc. **364** (2012), no. 11, 5945–5967.

[Dad77] E.C. Dade, *Remarks on isomorphic blocks*, J. Algebra **45** (1977), 254–258.

[Erd77] K. Erdmann, *Principal blocks of groups with dihedral Sylow* 2*-subgroups*, Comm. Algebra **5** (1977), 665–694.

[Erd90] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Mathematics **1428**. Springer-Verlag, Berlin, 1990.

[Gor68] D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.

[GW62] D. Gorenstein, J.H. Walter, *On finite groups with dihedral Sylow 2-subgroups*, Illinois J. Math. **6** (1962), 553–593.

[GW65] D. Gorenstein, J.H. Walter, *The characterization of finite groups with dihedral Sylow* 2*-subgroups, I, II, III*, J. Algebra **2** (1965), 85–151, 218–270, 354–393.

[HK00] G. Hiss, R. Kessar, *Scopes reduction and Morita equivalence classes of blocks in finite classical groups*, J. Algebra **230** (2000), 378–423.

[HK05] G. Hiss, R. Kessar, *Scopes reduction and Morita equivalence classes of blocks in finite classical groups II*, J. Algebra **283** (2005), 522–563.

[IK17] H. Ishioka, N. Kunugi, *Brauer indecomposability of Scott modules*, J. Algebra **470** (2017), 441–449.

[Kes96] R. Kessar, *Blocks and source algebras for the double covers of the symmetric and alternating groups*, J. Algebra **186** (1996), 872–933.

[Kes00] R. Kessar, *Equivalences for blocks of the Weyl groups*, Proc. Amer. Math. Soc. **128** (2000), 337–346.

[Kes01] R. Kessar, *Source algebra equivalences for blocks of finite general linear groups over a fixed field*. Manusc. Math. **104** (2001), 145–162.

[Kes02] R. Kessar, *Scopes reduction for blocks of finite alternating groups*, Quart. J. Math. **53** (2002), 443–454.

[KKL15] R. Kessar, S. Koshitani, M. Linckelmann, *On the Brauer indecomposability of Scott modules*, Quart. J. Math. **66** (2015), 895–903.

[KKM11] R. Kessar, N. Kunugi, N. Mitsuhashi, *On saturated fusion systems and Brauer indecomposability of Scott modules*, J. Algebra **340** (2011), 90–103.

[Kos82] S. Koshitani, *A remark on blocks with dihedral defect groups in solvable groups*, Math. Z. **179** (1982), 401–406.

[KK02] S. Koshitani, N. Kunugi, *Broué's conjecture holds for principal 3-blocks with elementary abelian defect group of order 9*, J. Algebra **248** (2002), 575–604.

[KMN11] S. Koshitani, J. Müller, F. Noeske, *Broué's abelian defect group conjecture holds for the sporadic simple Conway group* Co_3, J. Algebra **348** (2011), 354–380.

[Lan83] P. Landrock, *Finite Group Algebras and their Modules*, London Math. Soc. Lecture Note Series **84**, Cambridge University Press, Cambridge, 1983.

[Lin94] M. Linckelmann, *A derived equivalence for blocks with dihedral defect groups*, J. Algebra **164** (1994), 244–255.

[Lin96] M. Linckelmann, *Stable equivalences of Morita type for self-injective algebras and p-groups*, Math. Z. **223** (1996), 87–100.

[Lin96b] M. Linckelmann, *The isomorphism problem for cyclic blocks and their source algebras*, Invent. Math. **12** (1996), 265–283.

[Lin01] M. Linckelmann, *On splendid derived and stable equivalences between blocks of finite groups*, J. Algebra **242** (2001), 819–843.

[Lin18] M. Linckelmann, *The Block Theory of Finite Group Algebras*, Cambridge Univ. Press, Cambridge, 2018, in press.

[Mur09] J. Murray, *Real subpairs and Frobenius-Schur indicators of characters in* 2*-blocks*, J. Algebra **322** (2009), 489–513.

[NT88] H. Nagao, Y. Tsushima, *Representations of Finite Groups*, Academic Press, New York, 1988.

[Ple83] W. Plesken, *Group Rings of Finite Groups Over p-adic Integers*, Lecture Notes in Math. **1026**, Springer-Verlag, Berlin, 1983.

[Pui82] L. Puig, *Une conjecture de finitude sur les blocs*, unpublished manuscript, 1982.

[Pui94] L. Puig, *On Joanna Scopes' criterion of equivalence for blocks of symmetric groups*, Algebra Colloq. **1** (1994), 25–55.

[Pui99] L. Puig, *On the Local Structure of Morita and Rickard Equivalences between Brauer Blocks*, Birkhäuser, Basel, 1999.

[Ric96] J. Rickard, *Splendid equivalences: Derived categories and permutation modules*, Proc. London Math. Soc. **72** (1996), 331–358.

[Rou01] R. Rouquier, *Block theory via stable and Rickard equivalences*, in "Modular Representation Theory of Finite Groups", edited by M.J. Collins et al., Walter de Gruyter, Berlin, 2001, pp.101–146.

[Ste51] R. Steinberg, The representations of $\mathrm{GL}(3,q)$, $\mathrm{GL}(4,q)$, $\mathrm{PGL}(3,q)$, and $\mathrm{PGL}(4,q)$. Canad. J. Math. **3** (1951), 225–235.

[Suz86] M. Suzuki, *Group Theory II* , Springer, Heidelberg, 1986.

[Thé85] J. Thévenaz *Relative projective covers and almost split sequences*, Comm. Algebra **13** (1985), 1535–1554.

[Thé95] J. Thévenaz, *G-Algebras and Modular Representation Theory*, Clarendon Press, Oxford, 1995.

[Tuv14] İ. Tuvay, *On Brauer indecomposability of Scott modules of Park-type groups*, J. Group Theory **17** (2014), 1071–1079.

[Web82] P. J. Webb, *The Auslander-Reiten quiver of a finite group*, Math. Z. **179** (1982), 97–121.

[WPT+11] R. Wilson, J. Thackray, R. Parker, F. Noeske, J. Müller, F. Lübeck, C. Jansen, G. Hiss, T. Breuer, *The Modular Atlas Project* http://www.math.rwth-aachen.de/~MOC. Version 08.04.2011.

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