ON THE SOURCE ALGEBRA EQUIVALENCE CLASS OF BLOCKS WITH CYCLIC DEFECT GROUPS, I

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Abstract. We investigate the source algebra class of a block with cyclic defect groups of the group algebra of a finite group. By the work of Linckelmann this class is parametrized by the Brauer tree of the block together with a sign function on its vertices and an endo-permutation module of a defect group. We prove that this endo-permutation module can be read off from the character table of the group. We also prove that this module is trivial for all cyclic p-blocks of quasisimple groups with a simple quotient which is a sporadic group, an alternating group, a group of Lie type in defining characteristic, or a group of Lie type in cross-characteristic for which the prime p is large enough in a certain sense.

1. Introduction

Consider a finite group G, an algebraically closed field k of characteristic $p > 0$, and a p-block $\mathbf{B}$ of kG with a non-trivial cyclic defect group D. Let D_1 denote the unique subgroup of D of order p and let $\mathbf{b}$ be the Brauer correspondent of $\mathbf{B}$ in $N_G(D_1)$. For further terminology used below, we refer to Section 2.

By Linckelmann's results [Lin96, Theorem 2.7], the block $\mathbf{B}$ is determined up to source algebra equivalence by three invariants:

(a) the Brauer tree $\sigma(\mathbf{B})$ of $\mathbf{B}$ with its planar embedding;

(b) the type function on $\sigma(\mathbf{B})$, which associates one of the signs $+$ or $-$ to the vertices of $\sigma(\mathbf{B})$ in such a way that adjacent vertices have different signs; and

(c) an indecomposable capped endo-permutation kD-module $W(\mathbf{B})$ isomorphic, by definition, to a kD-source of the simple $\mathbf{b}$-modules.

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We observe that D_1 acts trivially on $W(\mathbf{B})$ and recall that (a) determines the Morita equivalence class of $\mathbf{B}$, whereas it is necessary to add parameters (b) and (c) to determine the source algebra of $\mathbf{B}$ up to isomorphism of interior D-algebras.

In [HL20, Section 5], the authors explicitly determined the structure of the indecomposable trivial source modules belonging to a cyclic p-block $\mathbf{B}$. This involves all three invariants (a), (b) and (c) above, and in particular the module $W(\mathbf{B})$ in an essential way. The formulae are particularly simple provided $\mathbf{B}$ is uniserial or $W(\mathbf{B}) \cong k$.

It is known, nevertheless, that all indecomposable capped endo-permutation kD-modules on which D_1 acts trivially arise as a source of the simple $\mathbf{b}$-modules for some cyclic block $\mathbf{B}$; this follows from examples given by Dade in [Dad66] (reinterpreted in terms of Linckelmann's definition of $W(\mathbf{B})$). Dade's construction was later extended and generalized by Mazza in [Maz03]. However, all the examples provided by the methods of [Dad66] and Mazza [Maz03] arise from p-blocks of p-solvable groups, and such blocks are uniserial as k-algebras (i.e. all their projective indecomposable modules are uniserial).

It is therefore natural to ask whether it is possible to describe all source algebra equivalence classes of cyclic blocks arising in finite groups in function of the invariants (a), (b) and (c), and in particular which capped endo-permutation kD-modules can occur as $W(\mathbf{B})$ for cyclic blocks $\mathbf{B}$ which are not uniserial. These questions are the main motivation behind the present article. More precisely, we are going to address the following questions.

Questions 1.1.

(a) *Can we determine $W(\mathbf{B})$ from the character table of G?*

(b) *Can we determine $W(\mathbf{B})$ for all non-uniserial cyclic blocks $\mathbf{B}$ of finite groups?*

(c) *Can we determine $W(\mathbf{B})$ for all cyclic blocks $\mathbf{B}$ of quasisimple groups?*

(d) *Can we reduce Question* (b) *to Question* (c)*?*

In Part I of our paper, in Theorem 3.4, we give a positive answer to Question 1.1(a) for odd primes p. We also provide further criteria for $W(\mathbf{B})$ to be trivial. Then, in Section 4 we work towards reduction theorems. In particular, we prove the analogues of Feit's reduction theorems [Fei84] for the determination of all Brauer trees. Our results, however, do not give a complete reduction of Question (b) to quasisimple groups. Finally, in Section 6, we start the classification of the modules $W(\mathbf{B})$ (up to isomorphism) for the cyclic p-blocks of the quasisimple groups. We treat the quasisimple groups with a simple quotient which is a sporadic group, an alternating group, a group of Lie type in defining characteristic, or a finite group of Lie type for which p is large in a certain sense. In all these cases, it turns out that $W(\mathbf{B}) \cong k$. However, this is not true in general, although the possibilities for $W(\mathbf{B})$ are rather restricted. For example, the group $\mathrm{SU}_9(2)$ has a cyclic 3-block $\mathbf{B}$ with $|D| = 9$, such that $W(\mathbf{B}) = \Omega_{D/D_1}(k) \not\cong k$; for this notation see Subsection 2.3 below. The complete classification of the modules $W(\mathbf{B})$ arising from quasisimple group, which includes the above example, is fairly delicate and is postponed to Part II of our paper.

Finally, we notice that the methods used in Section 6 are of a group-theoretical nature and show that in the treated cases all the centralizers of all the non-trivial p-elements lying in a given cyclic defect group are equal; this result is of independent interest.

2. Notation and preliminaries

2.1. **General Notation.** Throughout this paper we let p be a prime number and G be a finite group of order divisible by p. We let $(K, \mathcal{O}, k)$ be a p-modular system, where $\mathcal{O}$ is a complete discrete valuation ring with field of fractions K of characteristic zero and algebraically closed residue field k of characteristic p. We also assume that K is large enough in the sense that K is a splitting field for all subgroups of G.

Whenever D denotes a cyclic p-subgroup of G of order p^l where l is a positive integer, then for each $0 \leq i \leq l$ we denote by D_i the cyclic subgroup of D of order p^i and we set $N_i := N_G(D_i)$.

Unless otherwise stated, kG-modules are assumed to be finitely generated left kG-modules and by a p-block of G we mean a block of kG. Given a subgroup $H \leq G$, we let k denote the trivial kH-module, we write $\mathrm{Res}^G_H(M)$ for the restriction of the kG-module M to H, and $\mathrm{Ind}^G_H(N)$ for the induction of the kH-module N to G. Given a normal subgroup U of G, we write $\mathrm{Inf}^G_{G/U}(M)$ for the inflation of the $k[G/U]$-module M to G. An analogous notation is used for characters. We let Ω denote the Heller operator. The canonical homomorphism $\pi : \mathcal{O} \longrightarrow \mathcal{O}/J(\mathcal{O}) = k$ induces a bijection between the blocks of $\mathcal{O}G$ and the blocks of kG; we simply denote by $\mathrm{Irr}_K(\mathbf{B})$ and $\mathrm{IBr}_p(\mathbf{B})$ the set of irreducible K-characters, respectively the set of irreducible Brauer characters, of the preimage of a p-block $\mathbf{B}$ of G under π.

We also recall that two p-blocks $\mathbf{B}_1$ and $\mathbf{B}_2$ of finite groups with a common defect group D are called *source algebra equivalent* (or *Puig equivalent*, or *splendidly Morita equivalent*) if there is an isomorphism of interior D-algebras between a source algebra of $\mathbf{B}_1$ and a source algebra of $\mathbf{B}_2$. Equivalently, a source algebra equivalence between $\mathbf{B}_1$ and $\mathbf{B}_2$ is a Morita equivalence between $\mathbf{B}_1$ and $\mathbf{B}_2$ which is induced by a pair of bimodules with trivial sources; see [Lin94, Theorem 4.1].

2.2. **Radical p-subgroups.** By $O_p(G)$ we denote the largest normal p-subgroup of G. A p-subgroup $P \leq G$ is called *radical*, if $O_p(N_G(P)) = P$. A defect group of a p-block of G is a radical p-subgroup. The following lemmas are elementary and well-known, hence their proofs are omitted. Some of these results will only be used in Part II of our paper.

Lemma 2.1. *Let $P \leq G$ be an abelian radical p-subgroup. Then $O_p(C_G(P)) = P = O_p(Z(C_G(P)))$. Moreover, $N_G(P) = N_G(C_G(P))$.*

Lemma 2.2. *Let $Z \leq Z(G)$ and let P be a p-subgroup of G. For $H \leq G$, we write $\bar{H}$ for the image of H under the canonical epimorphism $G \to G/Z$. Then the following hold.*

(a) *If $p \nmid |Z|$, then $C_G(P)/Z = C_{\bar{G}}(\bar{P})$. If $O_p(Z) \leq P$, then $N_G(P)/Z = N_{\bar{G}}(\bar{P})$.*

(b) *Suppose that $N_G(P)/Z = N_{\bar{G}}(\bar{P})$. Then P is a radical p-subgroup of G if and only if $\bar{P}$ is a radical p-subgroup of $\bar{G}$.*

(c) *Suppose that $N \trianglelefteq G$ with $P \leq N$ such that $p \nmid [G:N]$. Then P is a radical p-subgroup of N if and only if P is a radical p-subgroup of G.*

2.3. **Endo-permutation modules.** Let P be a finite p-group. A kP-module W is called *endo-permutation* if $\mathrm{End}_k(W)$, endowed with the kP-module structure given by the conjugation action of G, is a permutation kP-module. Such an endo-permutation kP-module is called *capped* if it has an indecomposable direct summand with vertex P. In this case, $W \cong \mathrm{Cap}(W)^{\oplus m} \oplus X$, where $m \geq 1$ is an integer, all the indecomposable direct summands of X have vertices strictly contained in P, and $\mathrm{Cap}(W)$, called the *cap* of W, is, up to isomorphism, the unique indecomposable direct summand of W with vertex P. The *Dade group* of P, denoted $\mathbf{D}_k(P)$, is the set of isomorphism classes of indecomposable capped endo-permutation kP-modules with composition law induced by the tensor product over k, i.e.

$$[W_1] + [W_2] := [\mathrm{Cap}(W_1 \otimes_k W_2)]\,.$$

If $Q \leq P$, then we denote by $\Omega_{P/Q}$ the relative Heller operator with respect to Q. By definition, if W is a kP-module, then $\Omega_{P/Q}(W)$ is the kernel of a relative Q-projective cover of W. Moreover, $\Omega_{P/Q}(W)$ is an endo-permutation kP-module if and only if W is. The usual

Heller operator is $\Omega = \Omega_{P/\{1\}}$. For a detailed introduction to endo-permutation modules we refer to [Lin18, Sections 7.3, 7.4, 7.8], [Thé95, Sections 28, 29] and the survey [Thé07, §3-§4].

2.4. **Blocks with cyclic defect groups.** From now on, if not otherwise specified, we denote by $\mathbf{B}$ a p-block of kG with a cyclic defect group $D \cong C_{p^l}$ with $l \geq 1$, and we let $\sigma(\mathbf{B})$ denote its Brauer tree, understood with its planar embedding. We let $\mathbf{b}$ be the Brauer correspondent of $\mathbf{B}$ in $N_G(D_1)$ and $\mathbf{c}$ be a p-block of $C_G(D_1)$ covered by $\mathbf{b}$. Then D is also a defect group of $\mathbf{b}$ and $\mathbf{c}$. The inertial index of $\mathbf{b}$ is equal to the inertial index of $\mathbf{B}$, whereas $\mathbf{c}$ is nilpotent, that is, of inertial index one and has a unique non-exceptional character. The Brauer tree $\sigma(\mathbf{b})$ is a star with exceptional vertex at its center (if any). In particular, both $\mathbf{b}$ and $\mathbf{c}$ are uniserial blocks, in the sense that their projective indecomposable modules are uniserial. Indeed, the structure of these modules is encoded in the Brauer tree; see [Alp86, Section 17]. The simple $\mathbf{b}$-modules and the simple $\mathbf{c}$-module have a common kD-source, which we denote by $W(\mathbf{B})$. Obviously, $W(\mathbf{B}) = W(\mathbf{b}) = W(\mathbf{c})$ and this is always an indecomposable capped endo-permutation kD-module, on which D_1 acts trivially, see [Lin18, Theorem 11.1.5].

3. On the determination of $W(\mathbf{B})$

Throughout this section, we assume that p is a prime and $\mathbf{B}$ is a p-block of kG with a cyclic defect group $D \cong C_{p^l}$ where $l \geq 1$, and associated capped endo-permutation kD-module $W(\mathbf{B})$. For further notation, see Section 2.

3.1. **Recognizing $W(\mathbf{B})$ from the character table.** Here we assume that p is odd. To begin with, we need to describe the values of the characters of the lifts to $\mathcal{O}$ of the indecomposable capped endo-permutation kD-modules. The latter modules were classified by Dade in [Dad78]. Up to isomorphism, these are precisely the modules of the form

$$W_D(\alpha_0, \dots, \alpha_{l-1}) := \Omega^{\alpha_0}_{D/D_0} \circ \Omega^{\alpha_1}_{D/D_1} \circ \cdots \circ \Omega^{\alpha_{l-1}}_{D/D_{l-1}}(k)$$

with $\alpha_i \in \{0, 1\}$ for each $0 \leq i \leq l-1$. With this notation, the addition in $\mathbf{D}_k(D)$ is given by

$$W_D(\alpha_0, \dots, \alpha_{l-1}) + W_D(\alpha'_0, \dots, \alpha'_{l-1}) = W_D(\alpha_0 + \alpha'_0, \dots, \alpha_{l-1} + \alpha'_{l-1}),$$

where the addition of the labels on the right hand side has to be taken modulo 2. This follows from [La12, Proposition 2.12.5]. Hence

$$\mathbf{D}_k(D) = \langle \Omega_{D/D_i}(k) \mid 0 \leq i \leq l-1 \rangle \cong (\mathbb{Z}/2)^l \,. \tag{1}$$

Furthermore, each indecomposable capped endo-permutation kD-module W lifts in a unique way to an endo-permutation $\mathcal{O}D$-lattice $\widetilde{W}$ such that the determinant of the underlying representation is the trivial character. It is then said that $\widetilde{W}$ has *determinant one*. Let ρ_W denote the K-character afforded by $\widetilde{W}$. Then, our assumptions on $(K, \mathcal{O}, k)$ ensure that for every $u \in D$, we have $\rho_W(u) \in \mathbb{Z} \setminus \{0\}$ and we define

$$\omega_W(u) := \begin{cases} +1 & \text{if } \rho_W(u) > 0\,, \\ -1 & \text{if } \rho_W(u) < 0\,. \end{cases}$$

See [Thé95, §28 and §52] for details.

Lemma 3.1. *Let* $[W] \in \mathbf{D}_k(D)$ *and set* $L := \Omega_{D/D_0}(W)$. *Then* $\omega_L(u) = -\omega_W(u)$ *for every* $u \in D \setminus \{1\}$.

Proof. Recall that $\Omega_{D/D_0} = \Omega$ is simply the Heller operator. Since D is cyclic, the projective cover of W is isomorphic to kD and we have a short exact sequence

$$0 \longrightarrow L \longrightarrow kD \longrightarrow W \longrightarrow 0\,.$$

Then, $\mathcal{O}D$ is a projective cover of $\widetilde{W}$. Now, as p is odd, any permutation $\mathcal{O}P$-lattice has determinant one (see e.g. [LT19, Lemma 3.3(a)]). Therefore, the kernel of the projective cover of $\widetilde{W}$ also has determinant one, and so we have a short exact sequence

$$0 \longrightarrow \widetilde{L} \longrightarrow \mathcal{O}D \longrightarrow \widetilde{W} \longrightarrow 0\,.$$

As projective characters vanish at p-singular elements, it follows that $\rho_W(u) + \rho_L(u) = 0$ for every $u \in D \setminus \{1\}$. The claim follows. $\square$

Lemma 3.2. *For each* $1 \leq i \leq l$ *let* u_i *be a generator of the subgroup* D_i *of* D. *Then, the map*

$$\begin{array}{rccc} \Psi_l : & \mathbf{D}_k(D) & \longrightarrow & \{\pm 1\}^l \\ & [W] & \mapsto & (\omega_W(u_1), \ldots, \omega_W(u_l)) \end{array}$$

is an isomorphism and independent of the choice of the generators $u_1, \ldots, u_l$. *Furthermore, if* $W = W_D(\alpha_0, \ldots, \alpha_{l-1})$ *and* $1 \leq i \leq l$, *then*

$$\omega_W(u_i) = \begin{cases} +1 & \text{if } \sum_{j=0}^{i-1} \alpha_j \equiv 0 \pmod 2, \\ -1 & \text{if } \sum_{j=0}^{i-1} \alpha_j \equiv 1 \pmod 2. \end{cases}$$

Proof. We proceed by induction on l. So first assume that $l = 1$. Then $D \cong C_p$ and Equation (1) at the beginning of this subsection yield $\mathbf{D}_k(D) = \{[k], [\Omega_{D/D_0}(k)]\}$, where $k = W_D(\alpha_0)$ with $\alpha_0 = 0$ and $\Omega_{D/D_0}(k) = W_D(\alpha_0)$ with $\alpha_0 = 1$. Clearly, $\rho_k = 1_D$ is the trivial character and so $\omega_k(u) = 1$ for each $u \in D \setminus \{1\}$, whereas for $L :=$

$\Omega_{D/D_0}(k)$ we have $\omega_L(u) = -1$ for each $u \in D \setminus \{1\}$ by Lemma 3.1. Hence $\Psi_l([k]) = (+1)$, $\Psi_l([\Omega_{D/D_0}(k)]) = (-1)$ and all assertions hold in this case.

Next, assume that $l > 1$ and assume that the assertions hold for $l-1$. For $u \in D$, write $\overline{u} := uD_1 \in D/D_1 \cong C_{p^{l-1}}$, so that $\overline{u_i}$ is a generator of the cyclic subgroup of order p^{i-1} of D/D_1 for each $2 \le i \le l$.

According to [Thé95, Exercise (28.3)], we can decompose $\mathbf{D}_k(D)$ into the disjoint union

$$\mathbf{D}_k(D) = \mathrm{Inf}^D_{D/D_1}\big(\mathbf{D}_k(D/D_1)\big) \sqcup \Omega_{D/D_0}\Big(\mathrm{Inf}^D_{D/D_1}\big(\mathbf{D}_k(D/D_1)\big)\Big).$$

In other words, if $[W] \in \mathbf{D}_k(D)$, then there exist $[V] \in \mathbf{D}_k(D/D_1)$ and $\alpha_0 \in \{0,1\}$ such that

$$W \cong \Omega^{\alpha_0}_{D/D_0}\big(\mathrm{Inf}^D_{D/D_1}(V)\big)\,.$$

Now, on the one hand, if $\alpha_0 = 0$, then D_1 acts trivially on W. Thus, for every $u \in D$ we have $\rho_W(u) = \rho_V(\overline{u})$, implying that $\omega_W(u) = \omega_V(\overline{u})$. In particular, for every $u \in D_1 \setminus \{1\}$ we have $\rho_W(u) = \dim_k(V)$ and hence $\omega_W(u) = +1$. On the other hand, if $\alpha_0 = 1$, then $\omega_W(u) = -\omega_V(\overline{u})$ for every $u \in D \setminus \{1\}$ by Lemma 3.1. In particular, $\omega_W(u) = -1$ for every $u \in D_1 \setminus \{1\}$. So, in all cases, $\omega_W(u_1)$ is independent of the choice of u_1. It follows that we can identify Ψ_l with the map

$$\begin{array}{ccc} \mathbf{D}_k(D) & \longrightarrow & \{\pm1\} \times \{\pm1\}^{l-1} \\ [\mathrm{W}] & \mapsto & \big(\omega_W(u_1), \Psi_{l-1}([V])\big) \end{array}$$

where

$$\Psi_{l-1} : \mathbf{D}_k(D/D_1) \longrightarrow \{\pm1\}^{l-1}, [V] \mapsto (\omega_V(\overline{u_2}), \ldots, \omega_V(\overline{u_l}))\,.$$

By the induction hypothesis Ψ_{l-1} is bijective and independent of the choice of the generators $u_2, \ldots, u_l$. Therefore, so is Ψ_l by the above argument.

Finally, observe that if $W = W_D(\alpha_0, \ldots, \alpha_{l-1})$, then we have $V = W_{D/D_1}(\alpha_1, \ldots, \alpha_{l-1})$. It was proved above that $\omega_W(u_1) = +1$ if $\alpha_0 = 0$ and $\omega_W(u_1) = -1$ if $\alpha_0 = 1$, whereas for each $2 \le i \le l$ the induction hypothesis and Lemma 3.1 imply that

$$\begin{aligned} \omega_W(u_i) = (-1)^{\alpha_0} \cdot \omega_V(\overline{u_i}) &= \begin{cases} (-1)^{\alpha_0} \cdot (+1) & \text{if } \sum_{j=1}^{i-1} \alpha_j \equiv 0 \pmod 2, \\ (-1)^{\alpha_0} \cdot (-1) & \text{if } \sum_{j=1}^{i-1} \alpha_j \equiv 1 \pmod 2 \end{cases} \\ &= \begin{cases} +1 & \text{if } \sum_{j=0}^{i-1} \alpha_j \equiv 0 \pmod 2, \\ -1 & \text{if } \sum_{j=0}^{i-1} \alpha_j \equiv 1 \pmod 2, \end{cases} \end{aligned}$$

as claimed. □

Lemma 3.3. *Set $W := W(\mathbf{B})$. Let ξ denote the non-exceptional character in $\mathrm{Irr}_K(\mathbf{c})$ and let $u \in D \setminus \{1\}$. Then, $\xi(u)$ is a non-zero integer and the following assertions hold:*

(i) $\xi(u) > 0$ *if and only if* $\omega_W(u) = +1$*;*

(ii) $\xi(u) < 0$ *if and only if* $\omega_W(u) = -1$.

Moreover, ξ is constant on $D_i \setminus D_{i-1}$ for every $1 \leq i \leq l$.

Proof. As $\mathbf{c}$ is a nilpotent block, there is a Morita equivalence between $\mathbf{c}$ and kD. This equivalence can be lifted to a Morita equivalence over $\mathcal{O}$, which, by [Thé95, Theorem (52.8)(a)], in turn induces a unique character bijection $\mathrm{Irr}_K(\mathbf{c}) \longrightarrow \mathrm{Irr}_K(D)$ determining the generalized decomposition numbers of $\mathbf{c}$. It then follows from [Thé95, Theorem (52.8)(a)] again and Dade's Theorem on cyclic blocks (see e.g. [Dor72, Theorem 68.1(8)]) that this bijection maps ξ to the trivial character 1_D, because 1_D is the unique p-rational element of $\mathrm{Irr}_K(D)$. Therefore, Brauer's second main theorem (see [Thé95, Theorem (43.4)]) together with [Thé95, Theorem (52.8)(a)] imply that

$$\xi(u) = \omega_W(u) \cdot m \,, \tag{2}$$

where m is a sum of degrees of Brauer characters, hence a positive integer. Hence $\xi(u)$ is a non-zero integer and satisfies (i) and (ii). The last claim also follows from (2) as ω_W is constant on $D_i \setminus D_{i-1}$ for every $1 \leq i \leq l$ by Lemma 3.2. □

We can now prove that the module $W(\mathbf{B})$ can be read off from the values of the non-exceptional characters in $\mathrm{Irr}_K(\mathbf{B})$ at the non-trivial elements of D.

In the proof of this result, we will need to consider some specific modules, called *hooks*. We refer to our paper [HL20] for details, but these modules can be thought of as being the indecomposable $\mathbf{B}$-modules lying on the rim of the stable Auslander-Reiten quiver of the block.

Theorem 3.4. *Let p be an odd prime. Let $\mathbf{B}$ be a p-block of kG with a cyclic defect group $D \cong C_{p^l}$ where $l \geq 1$. Set $W := W(\mathbf{B})$ and let χ be a non-exceptional character in $\mathrm{Irr}_K(\mathbf{B})$. For each $1 \leq i \leq l$, let $u_i \in D_i \setminus D_{i-1}$. Then, $\chi(u_i)$ is a non-zero integer for every $1 \leq i \leq l$ and the following assertions hold.*

(a) *If $\chi(u_1) > 0$, then for every $2 \leq i \leq l$, we have*
   (i) $\chi(u_i) > 0$ *if and only if* $\omega_W(u_i) = +1$*;*
   (ii) $\chi(u_i) < 0$ *if and only if* $\omega_W(u_i) = -1$*;*

(b) *If $\chi(u_1) < 0$, then for every $2 \leq i \leq l$, we have*
   (i) $\chi(u_i) > 0$ *if and only if* $\omega_W(u_i) = -1$*;*
   (ii) $\chi(u_i) < 0$ *if and only if* $\omega_W(u_i) = +1$.

Moreover, W is determined up to isomorphism by the character values $\chi(u_1), \dots, \chi(u_l)$.

Proof. First assume that $G = C_G(D_1)$, so that $\mathbf{B} = \mathbf{c}$. Let ξ denote the non-exceptional character in $\mathrm{Irr}_K(\mathbf{c})$. By Lemma 3.3, clearly, $\xi(u_i)$ is a non-zero integer for every $1 \le i \le l$ and $\xi(u_1) > 0$ as D_1 acts trivially on W by Clifford theory. Hence only (a) can happen in this case and (i) and (ii) are given by Lemma 3.3.

Next assume that $G = N_1$, so that $\mathbf{B} = \mathbf{b}$. If ψ is a non-exceptional K-character of $\mathbf{b}$, then ψ lies above ξ and

$$\mathrm{Res}^{N_1}_{C_G(D_1)}(\psi) = \sum_{i=1}^{t} {}^{g_i}\xi$$

for suitable elements $g_1, \dots, g_t \in N_1$. Thus, for any $u_i \in D_i \setminus D_{i-1}$, we have

$$\psi(u_i) = \sum_{j=1}^{t} \xi(g_j^{-1} u_i g_j)\,.$$

Now, if $g_j^{-1} u_i g_j$ is not conjugate to an element of D in $C_G(D_1)$, then $\xi(g_j^{-1} u_i g_j) = 0$ by Green's theorem on zeroes of characters; see [CR81, (19.27)]. If $g_j^{-1} u_i g_j$ is conjugate to an element u of D in $C_G(D_1)$, then $|u| = |u_i|$, and so $\xi(g_j^{-1} u_i g_j) = \xi(u_i)$ by Lemma 3.3. Therefore, Lemma 3.3 also implies that $\psi(u_i)$ is a non-zero integer as $\xi(u_i)$ is and we have

$$\psi(u_i) > 0 \quad \text{if and only if} \quad \xi(u_i) > 0, \tag{3}$$

and

$$\psi(u_i) < 0 \quad \text{if and only if} \quad \xi(u_i) < 0. \tag{4}$$

Again in this case only (a) can happen and (i) and (ii) follow from (a) and (b) of Lemma 3.3.

Assume now that G is arbitrary. Then χ is the K-character of a lift $\widehat{H}$ to $\mathcal{O}$ of a hook H of $\mathbf{B}$; see [HL20, §3.5 and §4.2]. (In fact, if the inertial index of $\mathbf{B}$ is at least 2, then all lifts of H afford χ.) Let $f(H)$ and $f(\widehat{H})$ denote the Green correspondents in N_1 of H and $\widehat{H}$, respectively. Then, $f(H)$ belongs to $\mathbf{b}$ and is also a hook of $\mathbf{b}$. By [HL20, Lemma 4.1], we have that $f(H)$ is simple if $\chi(u_1) > 0$, whereas $f(H)$ is the Heller-translate of a simple $\mathbf{b}$-module if $\chi(u_1) < 0$.

So, first assume that $\chi(u_1) > 0$. Then, by [HL20, Lemma 4.2 and Corollary 4.3], $f(\widehat{H})$ lifts $f(H)$ and affords a non-exceptional K-character ψ of $\mathbf{b}$ satisfying $\chi(u_i) = \psi(u_i)$ for every $u_i \in D_i \setminus D_{i-1}$ and every $1 \leq i \leq l$. Thus (i) and (ii) of (a) follow from (3) and (4) and Lemma 3.3.

Next, assume that $\chi(u_1) < 0$. Then, an analogous argument yields that a lift of $\Omega(f(H)) \cong f(\Omega(H))$ affords a non-exceptional K-character ψ of $\mathbf{b}$ satisfying $\chi(u_i) = -\psi(u_i)$ for every $u_i \in D_i \setminus D_{i-1}$ and every $1 \leq i \leq l$, since projective characters vanish at p-singular elements. Thus (i) and (ii) of (b) follow from (3) and (4) and Lemma 3.3.

Finally, by Lemma 3.2, up to isomorphism W is uniquely determined by the signs $\omega_W(u_1), \ldots, \omega_W(u_l)$, where as seen above $\omega_W(u_1) = +1$ by definition of W. Hence, the last claim is immediate from (a) and (b). □

As a consequence, we can express $W(\mathbf{B})$ as $W(\mathbf{B}) = W_D(\alpha_0, \ldots, \alpha_{l-1})$ where the integers $\alpha_0, \ldots, \alpha_{l-1} \in \{0, 1\}$ are determined recursively using the formulae of Lemma 3.2.

Corollary 3.5. *For each $1 \leq i \leq l$, let $u_i \in D_i \setminus D_{i-1}$. Then, $W(\mathbf{B})$ is trivial if and only if there exists a non-exceptional character χ in $\mathrm{Irr}_K(\mathbf{B})$ such that either $\chi(u_i) > 0$ for all $1 \leq i \leq l$ or $\chi(u_i) < 0$ for all $1 \leq i \leq l$.*

Proof. The trivial kD-module affords the trivial character of D, implying that $\omega_k(u_i) = +1$ for every $1 \leq i \leq l$. Hence the claim follows from the bijection Ψ_l in Lemma 3.2 and Theorem 3.4. □

3.2. **Criteria for the triviality of $W(\mathbf{B})$.** Here, p is an arbitrary prime. We provide further theoretical criteria for the triviality of $W(\mathbf{B})$ not involving character theory.

Lemma 3.6. *In each of the following cases $W(\mathbf{B}) \cong k$.*
(a) *If $\mathbf{B}$ is the principal block of kG;*
(b) *if $C_G(D) = C_G(D_1)$ or $N_G(D) = N_G(D_1)$;*
(c) *if $p = 2$ and $|D| = 4$.*

Proof. (a) If $\mathbf{B}$ is the principal block of kG, then its Brauer correspondent $\mathbf{b}$ in N_1 is the principal block of kN_1. Hence the trivial kN_1-module is a simple $\mathbf{b}$-module. It follows that $W(\mathbf{B})$ is trivial.

(b) Since D is normal in $C_G(D)$ and in $N_G(D)$ it follows from Clifford theory that D acts trivially on the unique simple $kC_G(D)$-module and on the simple $kN_G(D)$-modules, hence they are trivial source modules. As we assume that $C_G(D) = C_G(D_1)$ or $N_G(D) = N_G(D_1)$, it follows that $W(\mathbf{B})$ is trivial.

(c) There are only two indecomposable capped endo-permutation modules of kD, namely k and $\Omega(k)$, where Ω denotes the Heller operator. But D_1 does not act trivially on $\Omega(k)$, hence $W(\mathbf{B}) \cong k$. □

Notice that Lemma 3.6(b) applies in particular if $D \leq Z(G)$ or $D = D_1$.

4. Reduction theorems

This section is devoted to Clifford theory with respect to the source algebra equivalence classes of blocks with cyclic defect groups. Throughout this section, G is a finite group and $\mathbf{B}$ a p-block of G with a non-trivial cyclic defect group D. For further notation refer to Subsection 2.4. Our aim is in particular to extend the results of [Fei84, Section 4]. When possible, we state these results in terms of source algebra equivalences, which obviously implies that the module $W(\mathbf{B})$ is left unchanged. Else we study the behavior of $W(\mathbf{B})$ under each of these results.

By definition, the kernel of $\mathbf{B}$ is the intersection of the kernels of the characters in $\mathrm{Irr}(\mathbf{B})$.

We begin with a well-known result on inflation; see, e.g. [Fei82, V, Lemma 4.3].

Lemma 4.1. *Let $H \trianglelefteq G$. Suppose that H is in the kernel of $\mathbf{B}$ and that $p \nmid |H|$. Put $\bar{G} := G/H$ and write $^- : kG \to k\bar{G}$ for the canonical epimorphism. Then $\mathbf{B}$ and its image $\bar{\mathbf{B}}$ in $k\bar{G}$ are source algebra equivalent.*

Proof. As $p \nmid |H|$, the image $\bar{\mathbf{B}}$ of $\mathbf{B}$ is isomorphic to $\mathbf{B}$. Moreover, the canonical epimorphism is an isomorphism of interior D-algebras between $\mathbf{B}$ and $\bar{\mathbf{B}}$. This gives our claim. □

[Fei84, Lemma 4.1] is the first Fong reduction, which can also be stated in terms of source-algebra equivalences.

Lemma 4.2. *Let $H \trianglelefteq G$ and let $\mathbf{B}_0$ be a p-block of H. Then there is a bijection between the p-blocks of G covering $\mathbf{B}_0$ and the p-blocks of $\mathrm{Stab}_G(\mathbf{B}_0)$ covering $\mathbf{B}_0$, under which corresponding blocks are source algebra equivalent.*

Proof. See [Lin18, Theorem 6.8.3]. □

The following lemma studies the behavior of $W(\mathbf{B})$ in the situation of [Fei84, Lemma 4.3 and Lemma 4.4].

Lemma 4.3. *Let $H \trianglelefteq G$ be a normal subgroup and let $\mathbf{B}_0$ be a G-stable p-block of H covered by $\mathbf{B}$. Assume that $D \cap H$ is a defect group of $\mathbf{B}_0$. If $\{1\} \lneqq D \cap H \leq D$, then $W(\mathbf{B}_0) \cong \mathrm{Cap}\left(\mathrm{Res}^D_{D\cap H}(W(\mathbf{B}))\right)$ as $k[D \cap H]$-modules. In particular $W(\mathbf{B}_0) \cong k$ provided $W(\mathbf{B}) \cong k$, and $W(\mathbf{B}_0) \cong W(\mathbf{B})$ provided $D \cap H = D$.*

Proof. Let $\mathbf{b}_0$ denote the Brauer correspondent of $\mathbf{B}_0$ in $N_H(D_1)$. By definition $W(\mathbf{B}_0) \cong W(\mathbf{b}_0)$ and $W(\mathbf{B}) \cong W(\mathbf{b})$, hence it suffices to prove that $W(\mathbf{b}_0) \cong \mathrm{Cap}\left(\mathrm{Res}^D_{D\cap H}(W(\mathbf{b}))\right)$.

Since D is cyclic and $\{1\} \lneq D\cap H$, we have $D_1 \leq D\cap H \leq D$ and the following inclusions of subgroups:

$$\begin{tikzcd}
G \arrow[dd, dash] \arrow[dr, dash] & & \\
& N_G(D_1) \arrow[dd, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & N_G(D\cap H) \arrow[dd, dash] \\
& N_H(D_1) \arrow[dr, dash] & \\
& & N_H(D\cap H)
\end{tikzcd}$$

where $N_H(D_1) = H \cap N_G(D_1) \lhd N_G(D_1)$. Therefore, by the generalized version of the Harris-Knörr correspondence (see [HK85, Theorem]) given in [Lin18, Theorem 6.9.3], there is a bijection

$$\mathrm{Bl}_p(N_G(D_1) \mid \mathbf{b}_0) \overset{\sim}{\longleftrightarrow} \mathrm{Bl}_p(G \mid \mathbf{B}_0)$$

given by the Brauer correspondence. Thus $\mathbf{b}$ covers $\mathbf{b}_0$. Moreover, it follows from the uniqueness of the Brauer correspondent that $\mathbf{b}_0$ is $N_G(D_1)$-stable (else $\mathbf{B}_0$ would not be G-stable).

Now, let M be a simple $\mathbf{b}$-module with kD-source S. By definition $W(\mathbf{b}) \cong S$, and $M \mid \mathrm{Ind}_D^{N_G(D_1)}(S)$. By Clifford theory, any indecomposable direct summand M_0 of $\mathrm{Res}^{N_G(D_1)}_{N_H(D_1)}(M)$ is a simple $\mathbf{b}_0$-module. Moreover,

$$M_0 \mid \mathrm{Res}^{N_G(D_1)}_{N_H(D_1)}(M) \mid \mathrm{Res}^{N_G(D_1)}_{N_H(D_1)} \mathrm{Ind}_D^{N_G(D_1)}(S),$$

and the Mackey formula yields

$$M_0 \mid \bigoplus_{x\in[N_H(D_1)\backslash N_G(D_1)/D]} \mathrm{Ind}^{N_H(D_1)}_{{}^xD\cap N_H(D_1)} \mathrm{Res}^{{}^xD}_{{}^xD\cap N_H(D_1)}({}^xS)\,. \tag{5}$$

Since $N_H(D_1) \lhd N_G(D_1)$, for any $x \in N_G(D_1)$ we have

$${}^xD \cap N_H(D_1) \cong D \cap {}^{x^{-1}}N_H(D_1) = D \cap N_H(D_1) = D\cap H\,.$$

Moreover, as S is a capped endo-permutation kD-module, it follows that $\mathrm{Res}^D_{D\cap N_H(D_1)}(S) = \mathrm{Res}^D_{D\cap H}(S) \cong (S_0)^{\oplus m} \oplus X_0$, where m is a positive integer, $S_0 := \mathrm{Cap}\left(\mathrm{Res}^D_{D\cap H}(S)\right)$ and all the indecomposable direct summands of X_0 have a vertex strictly contained in $D\cap N_H(D_1)$. Then, as M_0 has vertex $D\cap H$, it follows from (5) that there exists $x \in N_G(D_1)$ such that

$$M_0 \mid \mathrm{Ind}^{N_H(D_1)}_{{}^xD\cap N_H(D_1)}({}^xS_0)\,.$$

Therefore $({}^xD \cap N_H(D_1), {}^xS_0)$ is a vertex-source pair for M_0. Thus, identifying ${}^xD \cap N_H(D_1)$ with $D \cap H$, we obtain that $W(\mathbf{b}_0) \cong S_0$ as $k[D \cap H]$-modules, because the isomorphism class of an indecomposable $k[D \cap H]$-module is entirely determined by its k-dimension. The claim follows. $\square$

Corollary 4.4. *Let $H \trianglelefteq G$ be a normal subgroup and let $\mathbf{B}_0$ be a p-block of H covered by $\mathbf{B}$. If $D \leq H$, then $W(\mathbf{B}_0) \cong W(\mathbf{B})$.*

Proof. This is immediate from Lemmas 4.2 and 4.3. $\square$

The above corollary supplements [Fei84, Lemma 4.3]. Let us introduce some notation to make this more precise. Recall that $\sigma(\mathbf{B})$ denotes the Brauer tree of $\mathbf{B}$.

Definition 4.5. We call two cyclic p-blocks $\mathbf{B}'$ and $\mathbf{B}''$ *strongly similar*, if they have the same defect, if their non-embedded Brauer trees are similar in the sense of [Fei84, Section 2], and if $W(\mathbf{B}') \cong W(\mathbf{B}'')$, when the defect groups of $\mathbf{B}'$ and $\mathbf{B}''$ are identified.

It is clear that source algebra equivalent blocks are strongly similar. Assume now the hypothesis of Corollary 4.4. Then [Fei84, Lemma 4.3] and Corollary 4.4 imply that $\mathbf{B}$ and $\mathbf{B}_0$ are strongly similar. There is a situation in which these two blocks are even source algebra equivalent.

Lemma 4.6. *Assume the hypothesis of* Lemma 4.3 *and also that $D \leq H$. Put*

$$G[\mathbf{B}_0] := \{g \in G \mid g \text{ acts as inner automorphisms on } \mathbf{B}_0\}.$$

Suppose that $G[\mathbf{B}_0] = G$. Then $\mathbf{B}$ and $\mathbf{B}_0$ are source algebra equivalent.

Proof. This follows from [Kue90, Proposition 5 and Theorem 7]; see also [KKL12, Proposition 2.2]. $\square$

The following extends Linckelmann's version [Lin18, Theorem 6.8.13] of Fong's second reduction in the context of cyclic blocks. Important to us is the fact that this result describes a Morita equivalence induced by a bimodule with endo-permutation source. We give a proof which makes this endo-permutation module explicit.

Proposition 4.7. *Let $H \trianglelefteq G$ and let $\mathbf{B}_0$ be a p-block of H covered by $\mathbf{B}$. Suppose that $\mathbf{B}_0$ is G-stable and of defect 0. Let V be the simple $\mathbf{B}_0$-module.*

Then there is a natural action of D on V, which gives V the structure of a capped endo-permutation kD-module. Moreover, there is a central *extension $\widehat{G/H}$ of G/H with a center of order prime to p, and a p-block $\hat{\mathbf{B}}$ of $\widehat{G/H}$ with defect group $\hat{D}$, such that $\mathbf{B}$ and $\hat{\mathbf{B}}$ are basic Morita equivalent. In particular, D and $\hat{D}$ are isomorphic.*

If G/H is perfect, $\widehat{G/H}$ can be chosen to be perfect as well.

Let M and $\hat{M}$ denote a simple $\mathbf{B}$-module and a simple $\hat{\mathbf{B}}$-module corresponding under this Morita equivalence. Identify D with $\hat{D}$ and let $W(M)$ respectively $W(\hat{M})$ denote the D-sources of M respectively $\hat{M}$. Then $[W(M)] = [\mathrm{Cap}(V)] + [W(\hat{M})]$ in $\mathbf{D}_k(D)$,

Proof. As $\mathbf{B}_0$ has defect 0, first observe that $D \cap H = \{1\}$, so that we may identify D with a subgroup of G/H. Also $\mathbf{B}_0 \cong \mathrm{End}_k(V)$ as a k-algebra. Secondly, as $H \trianglelefteq G$, we have that $\mathbf{B}_0$ is a kD-module under the conjugation action of D, and as such is a direct summand of kH, hence a permutation kD-module. Moreover, the action of H on V extends in a unique way to an action of the semidirect product DH; see [Fei82, III, Corollary 3.16]. Thus, by definition, V is an endo-permutation kD-module.

We claim that the restriction of V to D contains an indecomposable direct summand V_0 with vertex D. Indeed, let $\iota \in Z(kH)$ denote the primitive idempotent with $\mathbf{B}_0 = \iota kH\iota$. As D is a defect group of the block $\mathbf{B}$ which covers $\mathbf{B}_0$, we have $\mathrm{Br}_D(\iota) \neq 0$. As $\mathbf{B}_0$ has a k-basis which is permuted by D, the image $\mathrm{Br}_D(\iota)$ is spanned by the D-fixed points of this basis. Thus $\mathbf{B}_0 \cong \mathrm{End}_k(V)$ has a trivial direct summand as a kD-module. It follows from [Ben98, Theorem 3.1.9], that V, as a kD-module, has an indecomposable direct summand V_0 of dimension prime to p. In particular, V_0 has vertex D and thus V is a capped endo-permutation kD-module.

By the Skolem–Noether theorem, any k-algebra automorphism of $\mathrm{End}_k(V)$ is inner. Thus every element $x \in G$ yields a unit $s_x \in \mathbf{B}_0^\times$ such that $xtx^{-1} = s_x t s_x^{-1}$ for all $t \in \mathbf{B}_0$. Moreover, for any $x, y \in H$, there is $\alpha(x, y) \in k^\times$ such that $s_{xy} = \alpha(x, y)s_x s_y$ with $\alpha(x, y) = 1$ whenever $x, y \in H$. We thus obtain an element $\alpha \in Z^2(G/N, k^\times)$. We may assume that the set of values of α lies in a finite field; see, e.g. [CR81, Lemma 11.38].

Let $k_\alpha[G/H]$ denote the twisted group algebra of G/H with respect to α, and let $\widetilde{G/H}$ be the finite central extension of G/H corresponding to $\alpha \in Z^2(G/H, k^\times)$. If G/H is perfect, put $\widehat{G/H} := [\widetilde{G/H}, \widetilde{G/H}]$, the commutator subgroup of $\widetilde{G/H}$. Otherwise, let $\widehat{G/H} := \widetilde{G/H}$. Then $\widehat{G/H}$ is a central extension of G/H, which is perfect if G/H

is, and $k[\widehat{G/H}]\epsilon \cong k_\alpha[G/H]$ as k-algebras for some central idempotent $\epsilon \in k[\widehat{G/H}]$; see [Thé95, Proposition 10.5] and [Sal08, Theorem 3.4].

There is an isomorphism

$$kG \otimes_{kH} \mathbf{B}_0 \to \mathbf{B}_0 \otimes_k k_\alpha[G/H]; \tag{6}$$

see [Lin18, Theorem 6.8.13]. In fact, (6) is an isomorphism of interior D-algebras, where the D-algebra structure on $k_\alpha[G/H]$ arises from the embedding of D into G/H; see [Sal08, Proposition 3.3]. We thus obtain an isomorphism

$$\Phi : kG \otimes_{kH} \mathbf{B}_0 \to \mathbf{B}_0 \otimes_k k[\widehat{G/H}]\epsilon \tag{7}$$

of interior D-algebras, where the D-algebra structure on $k[\widehat{G/H}]\epsilon$ arises from an embedding of D into $\widehat{G/H}$. Let $\hat{D} \leq \widehat{G/H}$ denote the image of this embedding. Now $\mathbf{B}$ is a block of $kG \otimes_{kH} \mathbf{B}_0$, and hence there is a block $\widehat{\mathbf{B}} = \widehat{\mathbf{B}}\epsilon$ of $k[\widehat{G/H}]$, such that

$$\Phi_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}_0 \otimes_k \widehat{\mathbf{B}} \tag{8}$$

is an isomorphism of interior D-algebras.

Identifying $\mathbf{B}$ and $\mathbf{B}_0 \otimes_k \widehat{\mathbf{B}}$ via (8), we obtain a Morita equivalence mod-$\widehat{\mathbf{B}} \to$ mod-$\mathbf{B}$, which is basic by [Pui99, Subsection 7.1 and Corollary 7.4]. This implies in particular, that the image $\hat{D}$ of D under $\Phi_{\mathbf{B}}$ is a defect group of $\hat{\mathbf{B}}$. Under this Morita equivalence, $\hat{M}$ is sent to $V \otimes_k \hat{M}$. Thus, as (8) is an isomorphism of interior D-algebras, this yields our claim about the sources of M and $\hat{M}$. □

5. Preliminaries on groups of Lie type

In order to investigate Question 1.1(c), we need to introduce some concepts and notation from the theory of finite groups of Lie type, where we largely follow the book [GeMa20] of Geck and Malle. Let r be a prime. Let $\mathbf{G}$ denote a connected reductive algebraic group over $\bar{\mathbb{F}}_r$, and let F be a Steinberg morphism of $\mathbf{G}$. If $\mathbf{H}$ is a closed subgroup of $\mathbf{G}$, we write $\mathbf{H}^\circ$ for its connected component, and if $\mathbf{H}$ is also F-stable, we write $H := \mathbf{H}^F$ for the finite group of F-fixed points of $\mathbf{H}$. By $\mathbf{G}^*$ we denote a connected reductive algebraic group dual to $\mathbf{G}^*$, equipped with a dual Steinberg morphism, which is also denoted by F. An F-stable Levi subgroup of $\mathbf{G}$ is called a *regular subgroup of* $\mathbf{G}$.

We record the following factorization lemma.

Lemma 5.1. *Let $\mathbf{G}$ denote a connected reductive algebraic group over $\bar{\mathbb{F}}_r$, and let F be a Steinberg morphism of $\mathbf{G}$.*

Then $\mathbf{G} = \mathbf{ZH}$*, with* $\mathbf{Z} = Z(\mathbf{G})^\circ$ *and* $\mathbf{H} = [\mathbf{G}, \mathbf{G}]$*. Moreover,* $|G| = |Z| \cdot |H|$*. In particular,* $ZH \trianglelefteq G$ *with* $[G\colon ZH] = |Z \cap H|$*, and* $|Z \cap H|$ *divides* $|Z(\mathbf{H})|$*.*

Proof. The factorization $\mathbf{G} = \mathbf{ZH}$ is standard. The multiplication map $\mathbf{Z} \times \mathbf{H} \to \mathbf{G}$ is a surjective, F-equivariant morphism of algebraic groups with a finite kernel, and thus $|G| = |Z| \cdot |H|$; see [GeMa20, Proposition 1.4.13]. □

Corollary 5.2. *Let the notation be as in* Lemma 5.1*. Let* p *be a prime dividing* $|Z|$ *but not* $|Z(\mathbf{H})|$*. Let* D *be a cyclic* p*-subgroup of* G *containing the Sylow* p*-subgroup* P *of* Z*. Then* $D = P$*.*

Proof. Lemma 5.1 implies that every Sylow p-subgroup of G is of the form $P \times Q$ for some Sylow p-subgroup Q of H. As $P \leq D$, this yields our claim. □

Choose an F-stable maximally split torus $\mathbf{T}$ of $\mathbf{G}$ and consider the character group $X := X(\mathbf{T})$, a free abelian group of finite rank. The action of F on $\mathbf{T}$ induces a linear map φ on $\mathbb{R} \otimes_{\mathbb{Z}} X$, which factors as $\varphi = q\varphi_0$ for a positive real number q and a linear map φ_0 of finite order; see [GeMa20, Proposition 1.4.19(b)]. Notice that q is the absolute value of all eigenvalues of φ, and thus q and φ_0 are uniquely determined by the pair $(\mathbf{G}, F)$. Moreover, $q^d = r^f$ for positive integers d, f; see [GeMa20, Proposition 1.4.19]. If q is an integer, it is a power of r. In [BrMa92, Section 1A], Broué and Malle define the concept of a *complete root datum*, also called a *generic finite reductive group*; for a slightly more general definition see [GeMa20, Definition 1.6.10]. Associated with the pair $(\mathbf{G}, F)$ is a generic finite reductive group $\mathbb{G}$; see [GeMa20, Example 1.6.11]. Conversely, if q is an integer, $(\mathbf{G}, F)$ can be constructed from $\mathbb{G}$ as explained in [BrMa92, Section 2]. Attached to $\mathbb{G}$ is a real polynomial, called the *order polynomial of* $\mathbb{G}$; for its definition see [BrMa92, Définition 1.9] or [GeMa20, Definition 1.6.10] and for its relevance see [BrMa92, Théorème 2.2] or [GeMa20, Remark 1.6.15]. By a slight abuse of language, the order polynomial of $\mathbb{G}$ is also called the order polynomial of $(\mathbf{G}, F)$.

We now cite the condition $(*)$ formulated in [Ma14, Subsection 2.1]. Assume that the positive real number q arising from $(\mathbf{G}, F)$ as above is an integer. Let p be a prime different from r. Then p satisfies condition $(*)$ with respect to $(\mathbf{G}, F)$, if there is a unique integer d such that $p \mid \Phi_d(q)$ and Φ_d divides the order polynomial of $(\mathbf{G}, F)$. Here, Φ_d denotes the dth cyclotomic polynomial over $\mathbb{Q}$. If p satisfies $(*)$, then p is odd and good for $\mathbf{G}$ and the Sylow p-subgroups of G are abelian; see [Ma14, Lemma 2.1 and Proposition 2.2].

Lemma 5.3. *Let* $\mathbf{G}$ *denote a connected reductive algebraic group over* $\bar{\mathbb{F}}_r$*, and let* F *be a Steinberg morphism of* $\mathbf{G}$*. Assume that the positive real number* q *associated to* $(\mathbf{G}, F)$ *as in* [GeMa20, Proposition 1.4.19] *is an integer (which then is a power of* r*). Let* p *be a prime different from* r *which satisfies condition* $(*)$ *given in* [Ma14]*. Assume in addition that* p *does not divide* $|Z(\mathbf{G})/Z(\mathbf{G})^\circ||Z(\mathbf{G}^*)/Z(\mathbf{G}^*)^\circ|$*.*

Let $D \leq G$ *denote a non-trivial cyclic radical* p*-subgroup of* G *of order* p^l*. Then* $C_{\mathbf{G}}(D) = C_{\mathbf{G}}(D_i)$ *for every* $1 \leq i \leq l$*.*

Proof. It suffices to prove the claim for $i = 1$. Put $\mathbf{L} := C_{\mathbf{G}}(D)$ and $\mathbf{L}_1 := C_{\mathbf{G}}(D_1)$. Then $\mathbf{L}$ and $\mathbf{L}_1$ are regular subgroups of $\mathbf{G}$ as p is good for $\mathbf{G}$ and does not divide $|Z(\mathbf{G}^*)/Z(\mathbf{G}^*)^\circ|$; see [GeHi91, Corollary 2.6]. Clearly, $\mathbf{L} \leq \mathbf{L}_1$ and $Z(\mathbf{L}_1) \leq Z(\mathbf{L})$ as $D \leq \mathbf{L}_1$. Also, $D = O_p(Z(L))$ by Lemma 2.1.

There is a surjection $Z(\mathbf{G})/Z(\mathbf{G})^\circ \to Z(\mathbf{L})/Z(\mathbf{L})^\circ$; see [Bon06, Proposition 4.2]. By assumption, $p \nmid |Z(\mathbf{G})/Z(\mathbf{G})^\circ|$, and hence $D \leq (Z(\mathbf{L})^\circ)^F$. Analogously, $D_1 \leq (Z(\mathbf{L}_1)^\circ)^F$.

Let d denote the order of q modulo p, and let $\mathbf{T}_d$ denote the Sylow Φ_d-torus of $Z(\mathbf{L})^\circ$; for the definition of a Sylow Φ_d-torus see [BrMa92, p. 254]. Since $O_p(Z(L)) = D$ is cyclic, the order of $\mathbf{T}_d^F$ equals $\Phi_d(q)$ and the rank of $\mathbf{T}_d$ equals $\deg(\Phi_d)$; see [BrMa92, Proposition 3.3].

From $Z(\mathbf{L}_1) \leq Z(\mathbf{L})$ we obtain $Z(\mathbf{L}_1)^\circ \leq Z(\mathbf{L})^\circ$. Let $\mathbf{T}'_d$ denote the Sylow Φ_d-torus of $Z(\mathbf{L}_1)^\circ$. Then $\mathbf{T}'_d \leq \mathbf{T}_d$, since $\mathbf{T}'_d$ lies in a Sylow Φ_d-torus of $Z(\mathbf{L})^\circ$, and $\mathbf{T}_d$ is the unique Sylow Φ_d-torus of $Z(\mathbf{L})^\circ$; see [BrMa92, Théorème 3.4(3)]. Clearly, $\mathbf{T}'_d$ is nontrivial, as $D_1 \leq (Z(\mathbf{L}_1)^\circ)^F$. In particular, the rank of $\mathbf{T}'_d$ is at least equal to $\deg(\Phi_d)$. It follows that $\mathbf{T}'_d = \mathbf{T}_d$.

Let d' be an integer such that $\Phi_{d'}$ divides the order polynomial of $(Z(\mathbf{L})^\circ, F)$. Then $\Phi_{d'}$ also divides the order polynomial of $(\mathbf{G}, F)$; see [BrMa92, Proposition 1.11]. By hypothesis, if $p \mid \mathbf{T}_{d'}^F$, then $d' = d$. It follows that $p \nmid [(Z(\mathbf{L})^\circ)^F : \mathbf{T}_d^F]$ and hence $D \leq \mathbf{T}_d^F$. Now $D \leq \mathbf{T}_d \leq Z(\mathbf{L})^\circ$, and thus $C_{\mathbf{G}}(D) \geq C_{\mathbf{G}}(\mathbf{T}_d) \geq C_{\mathbf{G}}(Z(\mathbf{L})^\circ) = \mathbf{L} = C_{\mathbf{G}}(D)$, as $\mathbf{L}$ is a regular subgroup of $\mathbf{G}$. We conclude that $C_{\mathbf{G}}(D) = C_{\mathbf{G}}(\mathbf{T}_d)$. Analogously, $C_{\mathbf{G}}(D_1) = C_{\mathbf{G}}(\mathbf{T}'_d)$. Our claim follows from $\mathbf{T}'_d = \mathbf{T}_d$. $\square$

The result of Lemma 5.3 can be generalized to the case when the centralizer of D is contained in a proper F-stable Levi subgroup of $\mathbf{G}$.

Corollary 5.4. *Let* $\mathbf{G}$ *denote a connected reductive algebraic group over* $\bar{\mathbb{F}}_r$*, and let* F *be a Steinberg morphism of* $\mathbf{G}$*. Assume that the positive real number* q *associated to* $(\mathbf{G}, F)$ *as in* [GeMa20, Proposition 1.4.19] *is an integer (which then is a power of* r*).*

Let p be a prime different from r and let $D \leq G$ denote a non-trivial cyclic p-subgroup of G of order p^l. Suppose that there is an F-stable Levi subgroup $\mathbf{L} \leq \mathbf{G}$ such $C_{\mathbf{G}}(D_1) \leq \mathbf{L}$, that p satisfies the condition $(*)$ with respect to $(\mathbf{L}, F)$ and that p does not divide $|Z(\mathbf{L})/Z(\mathbf{L})^\circ||Z(\mathbf{L}^*)/Z(\mathbf{L}^*)^\circ|$. Suppose finally that D is a radical subgroup of L. Then $C_{\mathbf{G}}(D) = C_{\mathbf{G}}(D_i)$ for every $1 \leq i \leq l$.

Proof. By Lemma 5.3 applied to $\mathbf{L}$, we obtain $C_{\mathbf{L}}(D) = C_{\mathbf{L}}(D_1)$. Hence $C_{\mathbf{G}}(D) = C_{\mathbf{L}}(D) = C_{\mathbf{L}}(D_1) = C_{\mathbf{G}}(D_1)$. This implies the assertion for all $1 \leq i \leq l$. □

6. The quasisimple groups

We are now going to address Question 1.1(c). Throughout this section, G denotes a quasisimple group, $S = G/Z(G)$ its simple quotient, and $\mathbf{B}$ is a p-block of G with a non-trivial cyclic defect group D.

6.1. **Some particular groups and special cases.** We begin with the case in which S is a sporadic simple group or the Tits simple group.

Proposition 6.1. *If S is one of the 26 simple sporadic groups or the Tits simple group ${}^2F_4(2)'$, then $W(\mathbf{B}) \cong k$.*

Proof. In all cases $D = D_1$, hence the claim follows from Lemma 3.6(b). □

Next, we consider the case in which S is an alternating group.

Proposition 6.2. *If S is the alternating group $\mathfrak{A}_n$ with $n \geq 5$, then $W(\mathbf{B}) \cong k$.*

Proof. First assume that $G = \mathfrak{A}_n$ ($n \geq 5$). The p-blocks of $\mathfrak{A}_n$, their weights and defect groups are, for example, described in [JK81, Subsection 6.1] and [Ols90, Section 4].

Let $\widehat{\mathbf{B}}$ be a block of $\mathfrak{S}_n$ of weight w covering $\mathbf{B}$. A defect group of $\widehat{\mathbf{B}}$ is conjugate to a Sylow p-subgroup of $\mathfrak{S}_{wp}$; see [JK81, Theorem 6.2.45]. Thus D is isomorphic to a Sylow p-subgroup of $\mathfrak{A}_{pw}$. If p is odd, this implies $w = 1$ and $D = D_1$, so the claim follows from Lemma 3.6(b). As the Sylow 2-subgroups of $\mathfrak{A}_{2w}$ are either trivial or non-cyclic, $\mathfrak{A}_n$ does not have 2-blocks with non-trivial cyclic defect groups.

Next assume that $|Z(G)| = 2$, i.e. $G = 2.\mathfrak{A}_n$. We let $\widetilde{\mathfrak{S}}_n$ denote a Schur covering group of $\mathfrak{S}_n$ containing G as a subgroup of index 2. For a description of the p-blocks of $2.\mathfrak{A}_n$ and $\widetilde{\mathfrak{S}}_n$ we refer to [Ols90], [BO97], [Hum86] and [Cab88].

Suppose that p is odd. In this case, it only remains to consider the faithful blocks, i.e. the spin blocks. There is a bijection between the spin blocks of $2.\mathfrak{A}_n$ of positive weight and the spin blocks of $\widetilde{\mathfrak{S}}_n$ of positive weight (given by covering), which preserves defect groups. Again, the spin blocks of $\widetilde{\mathfrak{S}}_n$ with cyclic defect groups are precisely the blocks of weight $w = 1$, which in turn have defect 1, since the defect groups are isomorphic to the Sylow p-subgroups of $\widetilde{\mathfrak{S}}_{pw}$. Hence the claim follows from Lemma 3.6(b).

If $p = 2$, then there is a bijection between the 2-blocks of $2.\mathfrak{A}_n$ and the 2-blocks of $\mathfrak{A}_n$, where defect groups are obtained by quotienting out the center; see [Fei82, V, Lemma 4.5]. As we have already observed above, the only 2-blocks of $\mathfrak{A}_n$ with cyclic defect groups are of defect 0. Hence $D \leq Z(G)$ and the claim follows from Lemma 3.6(b).

Finally, let G be an exceptional covering group of $\mathfrak{A}_n$ for $n \in \{6, 7\}$. If $p \geq 5$, we have $D = D_1$, as $p^2 \nmid |G|$. For $p \in \{2, 3\}$, we also get $D = D_1$ using [WTP$^+$]. Hence the claim follows from Lemma 3.6(b) in all cases. $\square$

From now on, we assume that S is a simple group of Lie type.

Proposition 6.3. *If S is a simple group of Lie type in characteristic p, then $W(\mathbf{B}) \cong k$.*

Proof. The Schur multiplier of S has order prime to p except for the 17 simple groups of Lie type listed in [GLS98, Table 6.1.3]. (These groups constitute 16 isomorphism types, as $\mathrm{PSL}_2(9) \cong \mathrm{Sp}_4(2)'$ is listed twice.) In any case, $G/O_p(Z(G))$ is a perfect central p'-extension of S. Let $\hat{G}$ denote the largest perfect central p'-extension of S. Then $G/O_p(Z(G))$ is a central quotient of $\hat{G}$.

If S is isomorphic to an alternating group, then the claim follows from Proposition 6.2. We can thus exclude the case $S \cong \mathrm{PSL}_2(9) \cong \mathrm{Sp}_4(2)' \cong \mathfrak{A}_6$. Then $\hat{G}$ is of the form $\hat{G} = \hat{\mathbf{G}}^F$ for a connected reductive algebraic group $\hat{\mathbf{G}}$ of characteristic p and a suitable Steinberg morphism F of $\hat{\mathbf{G}}$; see [GLS98, Definition 6.1.1(a) and Table 6.1.3]. By [Hum06, Theorem in Section 8.5], the defect groups of the p-blocks of $\hat{G}$ are either trivial or the Sylow p-subgroups of $\hat{G}$. Then, by Lemma 4.1, the same is true for the p-blocks of any central quotient of $\hat{G}$, so, in particular, it is also true for the p-blocks of $G/O_p(Z(G))$. Now, $\mathbf{B}$ determines a p-block of $G/O_p(Z(G))$ with defect group $D/O_p(Z(G))$; see [Fei82, V, Lemma 4.5]. Thus $D \leq Z(G)$ or $\hat{G}$ has a cyclic Sylow p-subgroup. In the former case, we obtain our assertion from Lemma 3.6(b). In the latter case, $S \cong \mathrm{PSL}_2(p)$ and $p \geq 5$. But then $D = D_1$ and the claim once more follows from Lemma 3.6(b). $\square$

We next consider those cases, where S has an exceptional Schur multiplier.

Proposition 6.4. *Assume that S is isomorphic to a simple group of Lie type with an exceptional Schur multiplier; see* [GLS98, Table 6.1.3]. *Then $W(\mathbf{B}) \cong k$.*

Proof. As in the proof of Proposition 6.3 we exclude the case $S \cong \mathrm{Sp}_4(2)'$.

Suppose first that p is odd. We claim that then $|D| = p$, which implies our assertion using Lemma 3.6(b). If $S \cong {}^2E_6(2)$ and $p \leq 5$, the claim can be checked with [GAP21], and in all other cases with [WTP$^+$].

Now assume that $p = 2$. If S has characteristic 2, the assertion follows from Proposition 6.3. Otherwise, S is isomorphic to $\mathrm{PSU}_4(3)$, $\Omega_7(3)$ or $G_2(3)$. In the first and third case, we find $|D| = p$ by inspecting [WTP$^+$]. It remains to consider the case $S \cong \Omega_7(3)$ which we handled with [GAP21]. We distinguish the cases $G = S$ and $G = \mathrm{Spin}_7(3) = 2.S$. If $G = S$, there is a unique 2-block of G with non-trivial cyclic defect group, and this has order 2. If $G = \mathrm{Spin}_7(3)$, there are three 2-blocks of G with non-trivial cyclic defect groups, two of order 2 and one of order 4. The claim follows from Lemma 3.6(b)(c). □

6.2. **Groups of Lie type and large primes.** In the considerations below we adopt the following common convention. Let $\varepsilon \in \{1, -1\}$. Then $\mathrm{PSL}_n^\varepsilon(q)$ denotes the projective special linear group $\mathrm{PSL}_n(q)$, if $\varepsilon = 1$, and the projective special unitary group $\mathrm{PSU}_n(q)$, if $\varepsilon = -1$. Analogous conventions are used for $\mathrm{SL}_n^\varepsilon(q)$ and $\mathrm{GL}_n^\varepsilon(q)$.

Proposition 6.5. *Let G denote a quasisimple group of Lie type with simple quotient S. Assume that S does not have an exceptional Schur multiplier and let $\hat{G}$ denote the universal covering group of G. Let p be an odd prime different from the defining characteristic r of S.*

Let $\mathbf{B}$ be a p-block of G with a non-trivial cyclic defect group D of order p^l. Then $C_G(D) = C_G(D_i)$ and $N_G(D) = N_G(D_i)$ for all $1 \leq i \leq l$ under any of the conditions in (a) *or* (b) *below. In particular, $W(\mathbf{B}) \cong k$ if these conditions are satisfied.*

(a) *Suppose that S is a classical group. Then $\hat{G}$ is one of $\mathrm{SL}_n(q)$ with $n \geq 2$, $\mathrm{SU}_n(q)$ with $n \geq 3$, $\mathrm{Sp}_n(q)$ with $n \geq 4$ even, $\mathrm{Spin}_n(q)$ with $n \geq 7$ odd, or $\mathrm{Spin}_n^\pm(q)$ with $n \geq 8$ even.*

Let d denote the order of q modulo p. If $\hat{G}$ is one of $\mathrm{SL}_n(q)$, $\mathrm{Sp}_n(q)$ or $\mathrm{Spin}_n^-(q)$, assume that $pd > n$. If $\hat{G} = \mathrm{Spin}_n(q)$ with n odd, assume that $pd > n-1$. If $\hat{G} = \mathrm{Spin}_n^+(q)$ with n even, assume that $pd > n-2$. Finally, if $\hat{G} = \mathrm{SU}_n(q)$, assume that $pd > 2n$.

(b) *Suppose that S is an exceptional group of Lie type, including the Suzuki and Ree groups. Assume that $p > 3$, and if S is of type E_8 assume that $p > 5$.*

Proof. It suffices to prove the first assertion, i.e. that $C_G(D) = C_G(D_i)$ for all $1 \leq i \leq l$. Indeed, Lemma 3.6(b) then implies that $W(\mathbf{B}) \cong k$. Moreover, $N_G(D) \leq N_G(D_i) \leq N_G(C_G(D_i)) = N_G(C_G(D)) = N_G(D)$ for all $1 \leq i \leq l$, where the last equality arises from Lemma 2.1.

Let us now prove the first assertion. To begin with, assume that S is a Suzuki group $S = {}^2B_2(2^{2m+1})$ for some $m \geq 2$, in which case $G = S$. It follows from the proof of [Suz62, Theorem 4], that the centralizer of every non-trivial p-element of S is a maximal torus. This implies our assertion. An analogous argument applies if S is a Ree group $S = {}^2G_2(3^{2m+1})$ for some $m \geq 1$, using [War66]. If $S = {}^2F_4(2^{2m+1})$ for some $m \geq 1$, we again have $G = S$. The conjugacy classes of S are determined in [Shi74]. Let $D = \langle t \rangle$; then t is a semisimple element of S and, by assumption, $3 \nmid |t|$. Now t is contained in one of the maximal tori $T(i)$, $i = 1, \ldots, 11$ of S, whose structure is given in [Shi74, p. 19]. Suppose first that t is a regular element of $T(i)$, i.e. $C_S(t) = T(i)$. As D is a radical p-subgroup of G, the Sylow p-subgroup of $T(i)$ is cyclic. Hence $i \in \{5, 9, 10, 11\}$. Indeed, a p-element in $T(2) \cup T(3) \cup T(4)$ is not regular, and the Sylow p-subgroups of $T(1)$, $T(6)$, $T(7)$ or $T(8)$ are not cyclic. From [Shi74, Table IV] we conclude that every non-trivial power of t also is regular in $T(i)$, proving our claim in the first case. Now assume that t is not a regular element. Then t is conjugate to one of t_j, with $j \in \{1, 2, 5, 7, 9\}$ in the notation of [Shi74, Table IV]. This table then shows that every non-trivial power of t has the same centralizer as t.

We will henceforth assume that S is not one of the Suzuki or Ree groups. By our assumption, p does not divide the order of the Schur multiplier of S. Thus, by Lemma 2.2(a), we may assume that $G = \hat{G}$ is the universal central extension of S. Thus there is a simple, simply connected algebraic group $\mathbf{G}$ over $\bar{\mathbb{F}}_r$, and a Steinberg morphism F of $\mathbf{G}$, such that $G = \mathbf{G}^F$. Moreover, $\mathbf{G}$ and F may be chosen so that the number q associated to $(\mathbf{G}, F)$ as in [GeMa20, Proposition 1.4.19(b)] is an integer. In fact it agrees with the number q introduced in the hypotheses (a). Now let d denote the order of q modulo p. Then $p \mid \Phi_d(q)$, and thus Φ_d divides the order polynomial of the generic finite reductive group $\mathbb{G}$ associated with $(\mathbf{G}, F)$; see [Ma07, Corollary 5.4]. Moreover, if d' is an integer such that $p \mid \Phi_{d'}(q)$, then $d' = p^j d$ for some non-negative integer j; see [Ma07, Lemma 5.2(a)]. The order polynomial of $\mathbb{G}$ can be read off from the order formulae for $\hat{G}$ given, for example in [GeMa20, Table 1.3]. We conclude that all hypotheses of Lemma 5.3 on $\mathbf{G}$, F and p are satisfied. Our claim follows from Lemma 5.3. $\square$

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