A Short Introduction to the Modular Representation Theory of Finite Groups

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Contents

Foreword

This text is a transcript of my lectures for the mini-course **Introduction to the Modular Representation Theory of Finite Groups** held at the **IKIU Summer School on Representation Theory 2022** and which took place from the 5th of September to the 7th of September 2022 in Qazin, Iran (3 x 80 minutes).

Together with the necessary theoretical foundations the main aim of this mini-course was to provide the participants with an introduction to the representation theory of finite groups from a module- and block-theoretic point of view.

The material presented here has been very much influenced by lectures and summer lectures available in conference proceedings or which I have followed myself as a student, doctoral student or young postdoc. In particular, I want to mention here:

- [Mal15] G. Malle, Darstellungstheorie, M.Sc. lecture, TU Kaiserslautern, WS 2015/16. [Unpublished]
- [Bro92] M. Broué, Equivalences of blocks of group algebras. Ottawa, 1992. `[MathSciNet]`
- [Kes07] R. Kessar, Introducton to block theory. EPFL: CIB 2005 & YAC 2012. `[MathSciNet]`
- [Kue18] B. Külshammer, Basic local representation theory. EPFL/CIB, 2016. `[MathSciNet]`
- [HKK10] G. Hiss, R. Kessar, B. Külshammer, An Introduction to the Representation Theory of Finite Groups. Aachen, 2010. [Unpublished]
- [Samb16] B. Sambale, Determination of block invariants. EPFL/CIB, 2016. [Unpublished, available from his webpage]

Acknowledgement:
First of all, I wish to thank the organisers of the *IKIU Summer School on Representation Theory* for inviting me to hold this mini-course and the efforts they invested in organising this event as a *presence* event. I believe, after 2 years of restrictions due to the pandemics, all participants immensely appreciated this opportunity to meet their contemporaries and hear about advances in current research problems.

Conventions

Unless otherwise stated, throughout these notes we make the following general assumptions:

- all groups considered are **finite**;
- all rings considered are **associative** and **unital**, i.e. possess a neutral element for the multiplication, denoted by 1;
- all modules considered are finitely generated **left** modules;

- R always denotes an associative ring with a 1;
- G always denotes a finite group;
- K always denotes a field of arbitrary characteristic;
- A always denotes a finite-dimensional K-algebra, which is split.
- $(F, \mathcal{O}, k)$ always denotes a splitting p-modular system for G and its subgroups, where F contains a primitive $\exp(G)$-th root of 1.

Monday, Chapter 1. Representations of Finite Groups

Representation theory of finite groups is originally concerned with the ways of writing a finite group G as a group of matrices, that is using group homomorphisms from G to the general linear group $\mathrm{GL}_n(K)$ of invertible $n \times n$-matrices with coefficients in a field K for some positive integer n. Thus, we shall first define representations of groups using this approach, and then translate such homomorphisms $G \longrightarrow \mathrm{GL}_n(K)$ into the language of module theory.

Notation. Throughout this chapter, unless otherwise specified, $\mathrm{char}(K) \geqslant 0$. Moreover, we assume that all KG-modules considered are finitely generated, so finite-dimensional when regarded as K-vector spaces.

1 Linear Representations of Finite Groups

To begin with, we review elementary definitions and examples about representations of finite groups.

Definition 1.1 (***K-representation, matrix representation***)

(a) A **K-representation** of G is a group homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$, where $V \cong K^n$ ($n \in \mathbb{Z}_{\geqslant 0}$) is a K-vector and $\mathrm{GL}(V) := \mathrm{Aut}_K(V)$.

(b) A **matrix representation** of G over K is a group homomorphism $X : G \longrightarrow \mathrm{GL}_n(K)$ ($n \in \mathbb{Z}_{\geqslant 0}$).

In both cases the integer n is called the **degree** of the representation.

(c) A K-representation (resp. a matrix representation) is called an **ordinary representation** if $\mathrm{char}(K) = 0$ (or more generally if $\mathrm{char}(K) \nmid |G|$), and it is called a **modular representation** if $\mathrm{char}(K) \mid |G|$.

Both concepts of a representation and of a matrix representation are closely related. Indeed, choosing a K-basis B of V, then we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, dashed, "\exists"'] & \mathrm{GL}(V) \arrow[d, leftrightarrow, "\cong"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{GL}_n(K)\,. \end{tikzcd}$$

Example 1

(a) The map

$$\begin{array}{rccl} \rho: & G & \longrightarrow & \mathrm{GL}(K) \cong K^\times \\ & g & \mapsto & \mathrm{Id}_K \leftrightarrow 1_K \end{array}$$

is a K-representation of G of degree 1, called **the trivial representation** of G.

(b) If X is a finite G-set, i.e. a finite set endowed with a left action $\cdot : G \times X \longrightarrow X$, and V is a K-vector space with basis $\{e_x \mid x \in X\}$, then

$$\begin{array}{rccl} \rho_X: & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_X(g) : V \longrightarrow V, e_x \mapsto e_{g\cdot x} \end{array}$$

is a K-representation of G of degree $|X|$, called the **permutation representation** associated with X.

Two particularly interesting examples are the following:

(1) if $G = S_n$ ($n \geqslant 1$) is the symmetric group on n letters, $X = \{1, 2, \ldots, n\}$, and the left action $\cdot : G \times X \longrightarrow X$ is given by the natural action of S_n then ρ_X is called **natural representation** of S_n;

(2) if $X = G$ and the left action $\cdot : G \times X \longrightarrow X$ is just the multiplication in G, then $\rho_X =: \rho_{\mathrm{reg}}$ is called the **regular representation** of G.

Definition 1.2 (*Homomorphism of representations, equivalent representations*)

Let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two K-representations of G.

(a) A K-homomorphism $\alpha : V_1 \longrightarrow V_2$ such that $\rho_2(g) \circ \alpha = \alpha \circ \rho_1(g)$ for each $g \in G$ is called a **homomorphism of representations** (or a G**-homomorphism**) between ρ_1 and ρ_2.

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & V_1 \arrow[d, "\alpha"] \\ V_2 \arrow[r, "\rho_2(g)"'] & V_2 \end{tikzcd}$$

(b) If, moreover, α is a K-isomorphism, then it is called an **isomorphism of representations** (or a G**-isomorphism**), and the K-representations ρ_1 and ρ_2 are called **equivalent** (or **similar**, or **isomorphic**). In this case we write $\rho_1 \sim \rho_2$.

Remark 1.3

(a) Equivalent representations have the same degree.

(b) Clearly $\sim$ is an equivalence relation.

(c) In consequence, it essentially suffices to study representations up to equivalence (as it essentially suffices to study groups up to isomorphism).

Definition 1.4 (*G-invariant subspace, irreducibility, subrepresentation*)

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a K-representation of G.

(a) A K-subspace $W \subseteq V$ is called G**-invariant** if

$$\rho(g)\big(W\big) \subseteq W \qquad \forall g \in G\,.$$

(In fact in this case the reverse inclusion holds as well, since for each $w \in W$ we can write $w = \rho(gg^{-1})(w) = \rho(g)\big(\rho(g^{-1})(w)\big) \in \rho(g)\big(W\big)$, hence $\rho(g)\big(W\big) = W$.)

(b) The representation ρ is called **irreducible** if it admits exactly two G-invariant K-subspaces, namely 0 and V itself; it is called **reducible** if it is not irreducible.

(c) If $0 \subsetneq W \subseteq V$ is a G-invariant K-subspace, then

$$\begin{array}{rccl} \rho_W : & G & \longrightarrow & \mathrm{GL}(W) \\ & g & \mapsto & \rho_W(g) := \rho(g)|_W : W \longrightarrow W \end{array}$$

is called a **subrepresentation** of ρ. (This is clearly again a K-representation of G.)

2 The Group Algebra and its Modules

We actually want to be able to see K-representations of a group G as *modules*.

Definition 2.1 (*Group algebra*)

The **group algebra of** G **over** K is the K-algebra KG whose elements are the K-linear combinations $\sum_{g\in G} \lambda_g g$ with $\lambda_g \in K\,\forall g \in G$, and addition and multiplication are given by

$$\sum_{g\in G} \lambda_g g + \sum_{g\in G} \mu_g g = \sum_{g\in G} (\lambda_g + \mu_g) g \quad \text{and} \quad \Big(\sum_{g\in G} \lambda_g g\Big) \cdot \Big(\sum_{h\in G} \mu_h h\Big) = \sum_{g,h\in G} (\lambda_g \mu_h) gh$$

respectively.

Remark 2.2

- $1_{KG} = 1_G$;
- $\dim_K(KG) = |G|$;
- KG is commutative if and only if G is an abelian group;
- as K is a field, KG is a left Artinian ring, so by Hopkins' Theorem a KG-module is finitely generated if and only if it admits a composition series.

Also notice that since G is a group, the map $KG \longrightarrow KG$ defined by $g \mapsto g^{-1}$ for each $g \in G$ is an anti-automorphism. It follows that any *left* KG-module M may be regarded as a *right* KG-module via the right G-action $m \cdot g := g^{-1} \cdot m$. Thus the sidedness of KG-modules is not usually an issue.

As KG is a K-algebra, we may of course consider modules over KG and we recall that any KG-module is in particular a K-vector space. Moreover, we adopt the following convention, which is automatically satisfied if K is a field.

Proposition 2.3

(a) Any K-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ of G gives rise to a KG-module structure on V, where the external composition law is defined by the map

$$\begin{array}{rccl} \cdot: & G \times V & \longrightarrow & V \\ & (g, v) & \mapsto & g \cdot v := \rho(g)(v) \end{array}$$

extended by K-linearity to the whole of KG.

(b) Conversely, every KG-module $(V, +, \cdot)$ defines a K-representation

$$\begin{array}{rccl} \rho_V : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_V(g) : V \longrightarrow V, v \mapsto \rho_V(g) := g \cdot v \end{array}$$

of the group G.

Example 2

Via Proposition 2.3 the trivial representation (Example 1(a)) corresponds to the so-called **trivial KG-module**, that is, K itself seen as a KG-module via the G-action

$$\begin{aligned} \cdot : G \times K &\longrightarrow K \\ (g, \lambda) &\longmapsto g \cdot \lambda := \lambda \end{aligned}$$

extended by K-linearity to the whole of KG.

Exercise 2.4

Prove that the regular representation ρ_{reg} of G defined in Exampale 1(b)(2) corresponds to the regular KG-module KG° via Proposition 2.3.

Convention: In the sequel, when no confusion is to be made, we drop the $\circ$-notation to denote the regular KG-module and simply write KG instead of KG°.

Remark 2.5 (*Dictionary*)

More generally, through Proposition 2.3, we may transport terminology and properties from KG-modules to K-representations and conversely.

This lets us build the following translation **dictionary**:

K-REPRESENTATIONS		KG-MODULES
K-representation of G	$\longleftrightarrow$	KG-module
degree	$\longleftrightarrow$	K-dimension
homomorphism of K-representations	$\longleftrightarrow$	homomorphism of KG-modules
equivalent K-representations	$\longleftrightarrow$	isomorphism of KG-modules
subrepresentation	$\longleftrightarrow$	KG-submodule
direct sum of representations $\rho_{V_1} \oplus \rho_{V_2}$	$\longleftrightarrow$	direct sum of KG-modules $V_1 \oplus V_2$
irreducible representation	$\longleftrightarrow$	simple (= irreducible) KG-module
the trivial representation	$\longleftrightarrow$	the trivial KG-module K
the regular representation of G	$\longleftrightarrow$	the regular KG-module KG°
completely reducible K-representation	$\longleftrightarrow$	semisimple KG-module (= completely reducible)
every K-representation of G is completely reducible	$\longleftrightarrow$	KG is semisimple
...		...

3 Operations on Groups and Modules

Next we see how to construct new KG-modules from old ones using standard module operations such as tensor products, Hom-functors, duality, or using subgroups or quotients of the initial group.

Remark 3.1 (*Tensors,* Hom*'s and duality*)

Let M and N be KG-modules.

(a) The tensor product $M \otimes_K N$ of M and N balanced over K becomes a KG-module via the **diagonal action** of G. In other words, the external composition law is defined by the G-action

$$\begin{array}{rccl} \cdot : & G \times (M \otimes_K N) & \longrightarrow & M \otimes_K N \\ & (g, m \otimes n) & \mapsto & g \cdot (m \otimes n) := gm \otimes gn \end{array}$$

extended by K-linearity to the whole of KG.

(b) The abelian group $\mathrm{Hom}_K(M, N)$ becomes a KG-module via the so-called **conjugation action** of G. In other words, the external composition law is defined by the G-action

$$\begin{array}{rccl} \cdot : & G \times \mathrm{Hom}_K(M, N) & \longrightarrow & \mathrm{Hom}_K(M, N) \\ & (g, f) & \mapsto & g \cdot f : M \longrightarrow N, m \mapsto (g \cdot f)(m) := g \cdot f(g^{-1} \cdot m) \end{array}$$

extended by K-linearity to the whole of KG.

(c) Specifying Definition 3.1 to $N = K$ yields a KG-module structure on the K-dual $M^* = \mathrm{Hom}_K(M, K)$, that is, M^* becomes a KG-module via the external composition law is defined by the map

$$\begin{array}{rccl} \cdot : & G \times M^* & \longrightarrow & M^* \\ & (g, f) & \mapsto & g \cdot f : M \longrightarrow K, m \mapsto (g \cdot f)(m) := f(g^{-1} \cdot m) \end{array}$$

extended by K-linearity to the whole of KG.

On the other hand, we may define new module structures from known ones for subgroups, overgroups and quotients. This leads to standard operations called *restriction*, *inflation*, and *induction*.

Remark 3.2

(a) If $H \leqslant G$ is a subgroup, then the inclusion $H \longrightarrow G, h \mapsto h$ can be extended by K-linearity to an injective algebra homomorphism $\iota : KH \longrightarrow KG, \sum_{h\in H} \lambda_h h \mapsto \sum_{h\in H} \lambda_h h$. Hence KH is a K-subalgebra of KG.

(b) Similarly, if $U \trianglelefteq G$ is a normal subgroup, then the quotient homomorphism $G \longrightarrow G/U$, $g \mapsto gU$ can be extended by K-linearity to an algebra homomorphism $\pi : KG \longrightarrow K[G/U]$.

It is clear that we can always perform changes of the base ring using the above homomorphism in order to obtain new module structures. This yields:

Definition 3.3 (*Restriction*)

Let $H \leqslant G$ be a subgroup. If M is a KG-module, then M may be regarded as a KH-module through a change of the base ring along $\iota : KH \longrightarrow KG$, which we denote by $\mathrm{Res}^G_H(M)$ or simply $M{\downarrow}^G_H$ and call the **restriction** of M from G to H.

Definition 3.4 (*Inflation*)

Let $U \trianglelefteq G$ be a normal subgroup. If M is a $K[G/U]$-module, then M may be regarded as a KG-module through a change of the base ring along $\pi : KG \longrightarrow K[G/U]$, which we denote by $\mathrm{Inf}^G_{G/U}(M)$ and call the **inflation** of M from G/U to G.

Lemma 3.5

(a) If $H \leqslant G$ and M_1, M_2 are two KG-modules, then $(M_1 \oplus M_2){\downarrow}^G_H = M_1{\downarrow}^G_H \oplus M_2{\downarrow}^G_H$. If $U \trianglelefteq G$ and M_1, M_2 are two $K[G/U]$-modules, then $\mathrm{Inf}^G_{G/U}(M_1 \oplus M_2) = \mathrm{Inf}^G_{G/U}(M_1) \oplus \mathrm{Inf}^G_{G/U}(M_2)$.

(b) **(Transitivity of restriction)** If $L \leqslant H \leqslant G$ and M is a KG-module, then $M{\downarrow}^G_H{\downarrow}^H_L = M{\downarrow}^G_L$.

(c) If $H \leqslant G$ and M is a KG-module, then $(M^*){\downarrow}^G_H \cong (M{\downarrow}^G_H)^*$. If $U \trianglelefteq G$ and M is a $K[G/U]$-module, then $\mathrm{Inf}^G_{G/U}(M^*) \cong (\mathrm{Inf}^G_{G/U} M)^*$.

A third natural operation comes from extending scalars from a subgroup to the initial group.

Definition 3.6 (*Induction*)

Let $H \leqslant G$ be a subgroup and let M be a KH-module. Regarding KG as a (KG, KH)-bimodule, we define the **induction** of M from H to G to be the left KG-module

$$M{\uparrow}^G_H = \mathrm{Ind}^G_H(M) := KG \otimes_{KH} M\,.$$

Example 3

(a) If $H = \{1\}$ and $M = K$, then $K\uparrow_{\{1\}}^{G} = KG \otimes_K K \cong KG$.

(b) **Transitivity of induction**: clearly $L \leqslant H \leqslant G$ and M is a KL-module, then $M\uparrow_L^G = (M\uparrow_L^H)\uparrow_H^G$ by the associativity of the tensor product.

First, we analyse the structure of an induced module in terms of the left cosets of H.

Remark 3.7

Writing $[G/H]$ $(= \{g_1, \ldots, g_{|G:H|}\})$ for a set of representatives of the left cosets, given a KH-module M, we have

$$KG \otimes_{KH} M = (\bigoplus_{g \in [G/H]} gKH) \otimes_{KH} M = \bigoplus_{g \in [G/H]} (gKH \otimes_{KH} M) = \bigoplus_{g \in [G/H]} (g \otimes M)\,,$$

where we set

$$g \otimes M := \{g \otimes m \mid m \in M\} \subseteq KG \otimes_{KH} M\,.$$

Clearly, each $g \otimes M \cong M$ as a K-space via the K-isomorphism $g \otimes M \longrightarrow M, g \otimes m \mapsto m$, so

$$\dim_K(\mathrm{Ind}_H^G(M)) = |G : H| \cdot \dim_K(M)\,.$$

Theorem 3.8 (*Adjunction / Frobenius reciprocity / Nakayama relations*)

Let $H \leqslant G$. Let N be a KG-module and let M be a KH-module. Then, there are K-isomorphisms:

(a) $\mathrm{Hom}_{KH}(M, \mathrm{Hom}_{KG}(KG, N)) \cong \mathrm{Hom}_{KG}(KG \otimes_{KH} M, N)$,
or in other words, $\mathrm{Hom}_{KH}(M, N\downarrow_H^G) \cong \mathrm{Hom}_{KG}(M\uparrow_H^G, N)\,;$

(b) $\mathrm{Hom}_{KH}(N\downarrow_H^G, M) \cong \mathrm{Hom}_{KG}(N, M\uparrow_H^G)\,.$

Proposition 3.9

Let $H \leqslant G$ be a subgroup. Let N be a KG-module and let M be a KH-module. Then, there are KG-isomorphisms:

(a) $(M \otimes_K N\downarrow_H^G)\uparrow_H^G \cong M\uparrow_H^G \otimes_K N$; and

(b) $\mathrm{Hom}_K(M, N\downarrow_H^G)\uparrow_H^G \cong \mathrm{Hom}_K(M\uparrow_H^G, N)$.

Finally, if H and L are subgroups of G, we wish to describe what happens if we induce a KL-module from L to G and then restrict it to H.

Now if M is a KL-module, we will also write ${}^gM$ for $g \otimes M$, which is a left $K({}^gL)$-module with

$$(glg^{-1}) \cdot (g \otimes m) = g \otimes lm$$

for each $l \in L$ and each $m \in M$.

Theorem 3.10 (*Mackey formula*)

Let $H, L \leqslant G$ and let M be a KL-module. Then, as KH-modules,

$$M\uparrow_L^G\downarrow_H^G \cong \bigoplus_{g\in[H\backslash G/L]} ({}^gM\downarrow^{^gL}_{H\cap {}^gL})\uparrow^H_{H\cap {}^gL} \ .$$

Proof: We need to examine KG seen as a (KH, KL)-bimodule (i.e. with left and right external laws given by multiplication in G). Since $G = \bigsqcup_{g\in[H\backslash G/L]} HgL$, we have

$$KG = \bigoplus_{g\in[H\backslash G/L]} K\langle HgL\rangle$$

as (KH, KL)-bimodule, where $K\langle HgL\rangle$ denotes the K-vector space with K-basis the double coset HgL. Also

$$K\langle HgL\rangle \cong KH \otimes_{K(H\cap {}^gL)} (g\otimes KL)\,,$$

where $hgl \in HgL$ corresponds to $h\otimes g\otimes l$. It follows that as left KH-modules we have

$$\begin{aligned} M\uparrow_L^G\downarrow_H^G \cong (KG\otimes_{KL} M)\downarrow_H^G &\cong \bigoplus_{g\in[H\backslash G/L]} K\langle HgL\rangle \otimes_{KL} M \\ &\cong \bigoplus_{g\in[H\backslash G/L]} KH\otimes_{K(H\cap {}^gL)} (g\otimes KL)\otimes_{KL} M \\ &\cong \bigoplus_{g\in[H\backslash G/L]} KH\otimes_{K(H\cap {}^gL)} (g\otimes M)\downarrow^{^gL}_{H\cap {}^gL} \\ &\cong \bigoplus_{g\in[H\backslash G/L]} ({}^gM\downarrow^{^gL}_{H\cap {}^gL})\uparrow^H_{H\cap {}^gL} \ . \end{aligned}$$

■

Monday. Chapter 2. Semisimplicity and Simplicity

The aim of this chapter is to study two important classes of modules over the group algebra, namely *simple modules* and *semisimple modules*. In particular, our first aim is to understand what the general theory of semisimple rings and the Artin-Wedderburn theorem bring to the theory of representations of finite groups over a field of arbitrary characteristic.

Notation. From now on, we let $\mathrm{Irr}(R) := \{\text{isomorphism classes of simple } R\text{-modules}\}$.

4 Schur's Lemma

Schur's Lemma is one of the most basic result, which lets us understand homomorphisms between *simple* modules, and, more importantly, endomorphisms of such modules. It is

Theorem 4.1 (*Schur's Lemma*)

(a) Let V, W be simple R-modules. Then:

(i) $\mathrm{End}_R(V)$ is a skew-field, and

(ii) if $V \ncong W$, then $\mathrm{Hom}_R(V, W) = 0$.

(b) If K is an algebraically closed field, A is a K-algebra, and V is a simple A-module, then

$$\mathrm{End}_A(V) = \{\lambda \,\mathrm{Id}_V \mid \lambda \in K\} \cong K\,.$$

Remark 4.2

In (b) the assumption that the field K is algebraically closed is in general too strong and we often replace this hypothesis by the hypothesis that the algebra A is **split**, meaning that

$$\mathrm{End}_A(S) \cong K \quad \text{for every simple } A\text{-module } S\,.$$

In this respect, the field K is a **splitting field** for G if the group algebra KG is split. This will be one of our standard assumptions.

From now on, we assume that K is a splitting field for G.

5 The Artin-Wedderburn Structure Theorem

The next step is to analyse semisimple rings and modules, sorting simple modules into isomorphism classes and relate these to a direct summand of the regular module.

Definition 5.1

If M is a semisimple R-module and S is a simple R-module, then the S-**homogeneous component** of M, denoted $S(M)$, is the sum of all simple R-submodules of M isomorphic to S.

Theorem 5.2 (*Wedderburn*)

If R is a semisimple ring, then the following assertions hold.

(a) If $S \in \mathrm{Irr}(R)$, then $S(R^\circ) \neq 0$. Furthermore, $|\mathrm{Irr}(R)| < \infty$.

(b) We have

$$R^\circ = \bigoplus_{S \in \mathrm{Irr}(R)} S(R^\circ)\,,$$

where each homogenous component $S(R^\circ)$ is a two-sided ideal of R and $S(R^\circ)T(R^\circ) = 0$ if $S \neq T \in \mathrm{Irr}(R)$.

(c) Each $S(R^\circ)$ is a simple left Artinian ring, the identity element of which is an idempotent element of R lying in $Z(R)$.

Remark 5.3

Remember that if R is a semisimple ring, then the regular module R° admits a composition series. Therefore it follows from the Jordan-Hölder Theorem that

$$R^\circ = \bigoplus_{S \in \mathrm{Irr}(R)} S(R^\circ) \cong \bigoplus_{S \in \mathrm{Irr}(R)} \bigoplus_{i=1}^{n_S} S$$

for uniquely determined integers $n_S \in \mathbb{Z}_{>0}$.

Theorem 5.4 (*Artin-Wedderburn*)

If R is a semisimple ring, then, as a ring,

$$R \cong \prod_{S \in \mathrm{Irr}(R)} M_{n_S}(D_S)\,,$$

where $D_S := \mathrm{End}_R(S)^{\mathrm{op}}$ is a division ring.

Let us now assume that $R = A$ is a split K-algebra.

We obtain the following Corollary to Wedderburn's and Artin-Wedderburn's Theorems.

Theorem 5.5

Assume A is semisimple and let $S \in \mathrm{Irr}(A)$ be a simple A-module. Then the following statements hold:

(a) $S(A^\circ) \cong M_{n_S}(K)$ and $\dim_K(S(A^\circ)) = n_S^2$;

(b) $\dim_K(S) = n_S$;

(c) $\dim_K(A) = \sum_{S \in \mathrm{Irr}(A)} \dim_K(S)^2$;

(d) $|\mathrm{Irr}(A)| = \dim_K(Z(A))$.

Exercise 5.6

Prove Thm. 5.5.

Corollary 5.7

Up to isomorphism, the number of simple A-modules is $|\mathrm{Irr}(A)| = \dim_K(Z(A/J(A)))$.

Proof: Since A and $A/J(A)$ have the same simple modules $|\mathrm{Irr}(A)| = |\mathrm{Irr}(A/J(A))|$. Moreover, the quotient $A/J(A)$ is J-semisimple, hence semisimple because finite-dimensional algebras are left Artinian rings. Therefore it follows from Theorem 5.5(d) that

$$|\mathrm{Irr}(A)| = |\mathrm{Irr}(A/J(A))| = \dim_K \left(Z(A/J(A))\right).$$

■

Corollary 5.8

If A is commutative, then any simple A-module has K-dimension 1.

Proof: First assume that A is semisimple. As A is commutative, $A = Z(A)$. Hence parts (d) and (c) of Theorem 5.5 yield

$$|\mathrm{Irr}(A)| = \dim_K(A) = \sum_{S \in \mathrm{Irr}(A)} \underbrace{\dim_K(S)^2}_{\geqslant 1},$$

which forces $\dim_K(S) = 1$ for each $S \in \mathrm{Irr}(A)$.
Now, if A is not semissimple, then again we use the fact that A and $A/J(A)$ have the same simple modules. Because $A/J(A)$ is semisimple and also commutative, the argument above tells us that all simple $A/J(A)$-modules have K-dimension 1. The claim follows. ■

Applying these results to the group algebra KG, we obtain for example that:

Corollary 5.9

There are only finitely many isomorphism classes of simple KG-modules.

Proof: The claim follows directly from Corollary 5.7. ■

Corollary 5.10

If G is an abelian group then any simple KG-module is one-dimensional.

Proof: Since KG is commutative the claim follows directly from Corollary 5.8. ■

Corollary 5.11

Let p be a prime number. If G is a p-group, and $\mathrm{char}(K) = p$, then the trivial module is the unique simple KG-module, up to isomorphism.

Proof: Because G is a p-group, we have $J(KG) = \{\sum_{g\in G} \lambda_g g \in KG \mid \sum_{g\in G} \lambda_g = 0\} =: I(KG)$ (the augmentation ideal (see definition in Exercise 6.3), so $KG/J(KG) \cong K$ as K-algebras. Now, as K is commutative, $Z(K) = K$, and it follows from Corollary 5.7 that

$$|\operatorname{Irr}(KG)| = \dim_K Z(KG/J(KG)) = \dim_K K = 1\,.$$

■

6 Semisimplicity of the Group Algebra

The semisimplicity of the group algebra depends on both the characteristic of the field and the order of the group. This is Maschke's Theorem and its converse.

Theorem 6.1 (*Maschke*)

If $\mathrm{char}(K) \nmid |G|$, then KG is a semisimple K-algebra.

Example 4

If $K = \mathbb{C}$ is the field of complex numbers, then $\mathbb{C}G$ is a semisimple $\mathbb{C}$-algebra, since $\mathrm{char}(\mathbb{C}) = 0$.

It turns out that the converse to Maschke's theorem also holds. This follows from elementary properties of the augmentation ideal.

Theorem 6.2 (*Converse of Maschke's Theorem*)

If KG is a semisimple K-algebra, then $\mathrm{char}(K) \nmid |G|$.

This result can be proved using the Artin-Wedderburn Theorem and elementary properties of augmentation ideal through the following exercices.

Exercise 6.3 (*The augmentation ideal*)

The map $\varepsilon : KG \longrightarrow K, \sum_{g\in G} \lambda_g g \mapsto \sum_{g\in G} \lambda_g$ is an algebra homomorphism, called **augmentation homomorphism (or map)**. Its kernel $\ker(\varepsilon) =: I(KG)$ is an ideal, called the **augmentation ideal** of KG. Prove that:

(a) $I(KG) = \{\sum_{g\in G} \lambda_g g \in KG \mid \sum_{g\in G} \lambda_g = 0\} = \mathrm{ann}_{KG}(K)$ and $I(KG) \supseteq J(KG)$;

(b) $KG/I(KG) \cong K$ as K-algebras;

(c) $I(KG)$ is a free K-vector space of dimension $|G|$-1 with K-basis $\{g - 1 \mid g \in G\backslash\{1\}\}$.

Exercise 6.4 (*Proof of the Converse of Maschke's Theorem.*)

Assume K is a field of positive characteristic p with $p \mid |G|$. Set $T := \langle \sum_{g\in G} g \rangle_K$.

(a) Prove that we have a series of KG-submodules given by $KG^\circ \supsetneq I(KG) \supseteq T \supsetneq 0$.

(b) Deduce that KG° has at least two composition factors isomorphic to the trivial module K.

(c) Deduce that KG is not a semisimple K-algebra.

Corollary 6.5

If $\mathrm{char}(K) \nmid |G|$, then $|G| = \sum_{S\in\mathrm{Irr}(KG)} \dim_K(S)^2$.

Proof: Since $\mathrm{char}(K) \nmid |G|$, the group algebra KG is semisimple by Maschke's Theorem. Thus

$$\sum_{S\in\mathrm{Irr}(KG)} \dim_K(S)^2 = \dim_K(KG) = |G|\,.$$

■

7 Clifford Theory

We now turn to *Clifford's theorem*, which we present in a weak and a strong form. Broadly speaking, Clifford theory is a collection of results about induction and restriction of simple modules from/to normal subgroups.

Theorem 7.1 (*Clifford's Theorem, weak form*)

If $U \trianglelefteq G$ is a normal subgroup and S is a simple KG-module, then $S\downarrow_U^G$ is semisimple.

Theorem 7.2 (*Clifford's Theorem, strong form*)

Let $U \trianglelefteq G$ be a normal subgroup and let S be a simple KG-module. Then we may write

$$S\downarrow_U^G = S_1^{a_1} \oplus \cdots \oplus S_r^{a_r}$$

where $r \in \mathbb{Z}_{>0}$ and $S_1, \ldots, S_r$ are pairwise non-isomorphic simple KU-modules, occurring with multiplicities $a_1, \ldots, a_r$ respectively. Moreover, the following statements hold:

(i) the group G permutes the homogeneous components of $S\downarrow_U^G$ transitively;

(ii) $a_1 = a_2 = \cdots = a_r$ and $\dim_K(S_1) = \cdots = \dim_K(S_r)$; and

(iii) $S \cong (S_1^{a_1})\uparrow_{H_1}^G$ as KG-modules, where $H_1 = \mathrm{Stab}_G(S_1^{a_1})$.

One application of Clifford's theory is for example the following Corollary:

Corollary 7.3

Assume K is a field of arbitrary characteristic. (Still splitting for G.) If p is a prime number and G is a p-group, then every simple KG-module has the form $X\uparrow_H^G$, where X is a 1-dimensional KH-module for some subgroup $H \leqslant G$.

Remark 7.4

This result is extremely useful, for example, to construct the complex character table of a p-group. Indeed, it says that we need look no further than induced linear characters. In general, a KG-module of the form $N\uparrow_H^G$ for some subgroup $H \leqslant G$ and some 1-dimensional KH-module is called **monomial**. A group all of whose simple $\mathbb{C}G$-modules are monomial is called an M-**group**. (By the above p-groups are M-groups.)