# Modules over Rings and Algebras: Basics

These notes provides you with a short recap of the notions of the theory of modules, which I will assume as known throughout this mini-course. The text is thought, so that you can refer to it if you have doubts about some elementary definitions and results, but proofs are omitted. For details I recommend Rotman's book below.

**References:**

[Rot10] J. J. Rotman. *Advanced modern algebra. 2nd ed.* Providence, RI: American Mathematical Society (AMS), 2010.

**Notation**: throughout these notes $R$ and $S$ denote rings. Unless otherwise specified, all rings are assumed to be *unital* and *associative*.

## A Modules, submodules, morphisms

**Definition A.1 (*Left $R$-module, right $R$-module, $(R,S)$-bimodule*)**

(a) A **left $R$-module** is an ordered triple $(M,+,\cdot)$, where $(M,+)$ is an abelian group and $\cdot : R \times M \longrightarrow M, (r,m) \mapsto r \cdot m$ is a binary operation such that the map

$$\begin{array}{rccl} \lambda : & R & \longrightarrow & \mathrm{End}(M) \\ & r & \mapsto & \lambda(r) := \lambda_r : M \longrightarrow M, m \mapsto r \cdot m \end{array}$$

is a ring homomorphism. The operation $\cdot$ is called a **scalar multiplication** or an **external composition law**.

(b) A **right $R$-module** is defined analogously using a scalar multiplication $\cdot : M \times R \longrightarrow M$, $(m,r) \mapsto m \cdot r$ on the right-hand side.

(c) An $(R,S)$**-bimodule** is an abelian group $(M,+)$ which is both a left $R$-module and a right $S$-module, and which satisfies the axiom

$$r \cdot (m \cdot s) = (r \cdot m) \cdot s \qquad \forall\, r \in R, \forall\, s \in S, \forall\, m \in M\,.$$

**Convention**: Unless otherwise stated, in this lecture we always work with left modules. When no confusion is to be made, we will simply write "$R$-module" to mean "left $R$-module", denote $R$-modules by their underlying sets and write $rm$ instead of $r \cdot m$.

Definitions/properties for/of right modules and bimodules are similar to those for left modules, hence in the sequel we omit them.

**Definition A.2 (*R-submodule*)**

An $R$-**submodule** of an $R$-module $M$ is a subgroup $U \leqslant M$ such that $r \cdot u \in U \ \forall\ r \in R, \forall\ u \in U$.

**Definition A.3 (*Morphisms*)**

A **(homo)morphism** of $R$-modules (or an $R$**-linear map**, or an $R$**-homomorphism**) is a map of $R$-modules $\varphi : M \longrightarrow N$ such that:

(i) $\varphi$ is a group homomorphism; and

(ii) $\varphi(r \cdot m) = r \cdot \varphi(m) \ \forall\ r \in R, \forall\ m \in M$.

Furthermore:

- An injective (resp. surjective) morphism of $R$-modules is sometimes called a **monomorphism** (resp. an **epimorphism**) and we often denote it with a *hook arrow* "$\hookrightarrow$" (resp. a *two-head arrow* "$\twoheadrightarrow$").

- A bijective morphism of $R$-modules is called an **isomorphism** (or an $R$**-isomorphism**), and we write $M \cong N$ if there exists an $R$-isomorphism between $M$ and $N$.

- A morphism from an $R$-module to itself is called an **endomorphism** and a bijective endomorphism is called an **automorphism** .

**Notation A.4**

We let ${}_R\mathbf{Mod}$ denote the category of left $R$-modules (with $R$-linear maps as morphisms), we let $\mathbf{Mod}_R$ denote the category of right $R$-modules (with $R$-linear maps as morphisms), and we let ${}_R\mathbf{Mod}_S$ denote the category of $(R, S)$-bimodules (with $(R, S)$-linear maps as morphisms).

**Remark A.5**

(a) It is easy to check that Definition A.1(a) is equivalent to requiring that $(M, +, \cdot)$ satisfies the following axioms:

**(M1)** $(M, +)$ is an abelian group;

**(M2)** $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$ for each $r_1, r_2 \in R$ and each $m \in M$;

**(M3)** $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$ for each $r \in R$ and all $m_1, m_2 \in M$;

**(M4)** $(rs) \cdot m = r \cdot (s \cdot m)$ for each $r, s \in R$ and all $m \in M$.

**(M5)** $1_R \cdot m = m$ for each $m \in M$.

In other words, modules over rings satisfy the same axioms as vector spaces over fields. Hence: Vector spaces over a field $K$ are $K$-modules, and conversely.

(b) Abelian groups are $\mathbb{Z}$-modules, and conversely.

(c) If the ring $R$ is commutative, then any right module can be made into a left module, and conversely.

(d) **Change of the base ring.**
If $\varphi : S \longrightarrow R$ is a ring homomorphism, then every $R$-module $M$ can be endowed with the structure of an $S$-module with external composition law given by

$$\cdot : S \times M \longrightarrow M, (s, m) \mapsto s \cdot m := \varphi(s) \cdot m \,.$$

(e) If $\varphi : M \longrightarrow N$ is a morphism of $R$-modules, then the kernel $\ker(\varphi) := \{m \in M \mid \varphi(m) = 0_N\}$ of $\varphi$ is an $R$-submodule of $M$ and the image $\mathrm{Im}(\varphi) := \varphi(M) = \{\varphi(m) \mid m \in M\}$ of $\varphi$ is an $R$-submodule of $N$.
If $M = N$ and $\varphi$ is invertible, then the inverse is the usual set-theoretic *inverse map* $\varphi^{-1}$ and is also an $R$-homomorphism.

**Notation A.6**

Given $R$-modules $M$ and $N$, we set $\mathrm{Hom}_R(M, N) := \{\varphi : M \longrightarrow N \mid \varphi \text{ is an } R\text{-homomorphism}\}$. This is an abelian group for the pointwise addition of maps:

$$\begin{array}{rccl} +: & \mathrm{Hom}_R(M,N) \times \mathrm{Hom}_R(M,N) & \longrightarrow & \mathrm{Hom}_R(M,N) \\ & (\varphi, \psi) & \mapsto & \varphi + \psi : M \longrightarrow N, m \mapsto \varphi(m) + \psi(m) \,. \end{array}$$

In case $N = M$, we write $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ for the set of endomorphisms of $M$ and $\mathrm{Aut}_R(M)$ for the set of automorphisms of $M$, i.e. the set of invertible endomorphisms of $M$.

**Lemma-Definition A.7 (*Quotients of modules*)**

Let $U$ be an $R$-submodule of an $R$-module $M$. The quotient group $M/U$ can be endowed with the structure of an $R$-module in a natural way via the external composition law

$$\begin{aligned} R \times M/U &\longrightarrow M/U \\ \big(r, m + U\big) &\longmapsto r \cdot m + U \end{aligned}$$

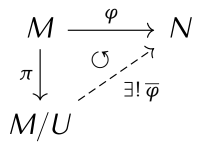
The canonical map $\pi : M \longrightarrow M/U, m \mapsto m + U$ is $R$-linear and we call it the **canonical** (or **natural**) **(ho)momorphism** or the **quotient (ho)momorphism**.

**Definition A.8 (*Cokernel, coimage*)**

Let $\varphi \in \mathrm{Hom}_R(M, N)$. The **cokernel** of $\varphi$ is the quotient $R$-module $\mathrm{coker}(\varphi) := N/\operatorname{Im}\varphi$, and the **coimage** of $\varphi$ is the quotient $R$-module $M/\ker\varphi$.

**Theorem A.9** (***The universal property of the quotient and the isomorphism theorems***)

(a) **Universal property of the quotient**: Let $\varphi : M \longrightarrow N$ be a homomorphism of $R$-modules. If $U$ is an $R$-submodule of $M$ such that $U \subseteq \ker(\varphi)$, then there exists a unique $R$-module homomorphism $\overline{\varphi} : M/U \longrightarrow N$ such that $\overline{\varphi} \circ \pi = \varphi$, or in other words such that the following diagram commutes:




Concretely, $\overline{\varphi}(m + U) = \varphi(m) \ \forall \ m + U \in M/U$.

(b) **1st isomorphism theorem**: With the notation of (a), if $U = \ker(\varphi)$, then

$$\overline{\varphi} : M/\ker(\varphi) \longrightarrow \operatorname{Im}(\varphi)$$

is an isomorphism of $R$-modules.

(c) **2nd isomorphism theorem**: If $U_1, U_2$ are $R$-submodules of $M$, then so are $U_1 \cap U_2$ and $U_1 + U_2$, and there is an isomorphism of $R$-modules

$$(U_1 + U_2)/U_2 \cong U_1/(U_1 \cap U_2)\,.$$

(d) **3rd isomorphism theorem**: If $U_1 \subseteq U_2$ are $R$-submodules of $M$, then there is an isomorphism of $R$-modules

$$(M/U_1)\big/(U_2/U_1) \cong M/U_2\,.$$

(e) **Correspondence theorem**: If $U$ is an $R$-submodule of $M$, then there is a bijection

$$\begin{array}{ccc} \{R\text{-submodules } X \text{ of } M \mid U \subseteq X\} & \longleftrightarrow & \{R\text{-submodules of } M/U\} \\ X & \mapsto & X/U \\ \pi^{-1}(Z) & \leftarrow\!\shortmid & Z\,. \end{array}$$

# B Free modules and projective modules

## Free modules

**Definition B.1** (***Generating set / R-basis / finitely generated/free R-module***)

Let $M$ be an $R$-module and let $X \subseteq M$ be a subset. Then:

(a) $M$ is said to be **generated by** $X$ if every element $m \in M$ may be written as an $R$-linear combination $m = \sum_{x \in X} \lambda_x x$, i.e. where $\lambda_x \in R$ is almost everywhere 0. In this case we write $M = \langle X \rangle_R$ or $M = \sum_{x \in X} Rx$.

(b) $M$ is said to be **finitely generated** if it admits a finite set of generators.

(c) $X$ is an $R$**-basis** (or simply a **basis**) if $X$ generates $M$ and if every element of $M$ can be written *in a unique way* as an $R$-linear combination $\sum_{x \in X} \lambda_x X$ (i.e. with $\lambda_x \in R$ almost everywhere 0).

(d) $M$ is called **free** if it admits an $R$-basis $X$, and $|X|$ is called the $R$**-rank** of $M$.
**Notation:** In this case we write $M = \bigoplus_{x \in X} Rx \cong \bigoplus_{x \in X} R$.

**Remark B.2**

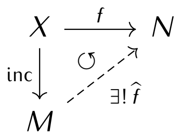
(a) **Warning**: If the ring $R$ is not commutative, then it is not true in general that two different bases of a free $R$-module have the same number of elements.

(b) Let $X$ be a generating set for $M$. Then, $X$ is a basis of $M$ if and only if $S$ is $R$-linearly independent.

(c) If $R$ is a field, then every $R$-module is free. ($R$-modules are $R$-vector spaces in this case!)

**Proposition B.3 (*Universal property of free modules*)**

Let $M$ be a free $R$-module with $R$-basis $X$. If $N$ is an $R$-module and $f : X \longrightarrow N$ is a map (of sets), then there exists a unique $R$-homomorphism $\widehat{f} : M \longrightarrow N$ such that the following diagram commutes:




We say that $\widehat{f}$ is obtained by **extending** $f$ **by** $R$**-linearity**.

**Proof:** Given an $R$-linear combination $\sum_{x \in X} \lambda_x x \in M$, set $\widehat{f}(\sum_{x \in X} \lambda_x x) := \sum_{x \in X} \lambda_x f(x)$. ■

**Proposition B.4 (*Properties of free modules*)**

(a) Every $R$-module $M$ is isomorphic to a quotient of a free $R$-module.

(b) If $P$ is a free $R$-module, then $\mathrm{Hom}_R(P, -)$ is an exact functor.

## Projective modules

**Proposition-Definition B.5 (*Projective module*)**

Let $P$ be an $R$-module. Then the following are equivalent:

(a) The functor $\mathrm{Hom}_R(P, -)$ is exact.

(b) If $\psi \in \mathrm{Hom}_R(M, N)$ is a surjective morphism of $R$-modules, then the morphism of abelian groups $\psi_* : \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N)$ is surjective.

(c) If $\pi \in \mathrm{Hom}_R(M, P)$ is a surjective morphism of $R$-modules, then $\pi$ splits, i.e., there exists $\sigma \in \mathrm{Hom}_R(P, M)$ such that $\pi \circ \sigma = \mathrm{Id}_P$.

(d) $P$ is isomorphic to a direct summand of a free $R$-module.

If $P$ satisfies these equivalent conditions, then $P$ is called **projective**.

**Example B.6**

(a) If $R = \mathbb{Z}$, then every submodule of a free $\mathbb{Z}$-module is again free (main theorem on $\mathbb{Z}$-modules).

(b) Let $e$ be an idempotent in $R$, that is $e^2 = e$. Then, $R \cong Re \oplus R(1-e)$ and $Re$ is projective but not free if $e \neq 0, 1$.

(d) A direct sum of modules $\bigoplus_{i\in I} P_i$ is projective if and only if each $P_i$ is projective.

## C Direct products and direct sums

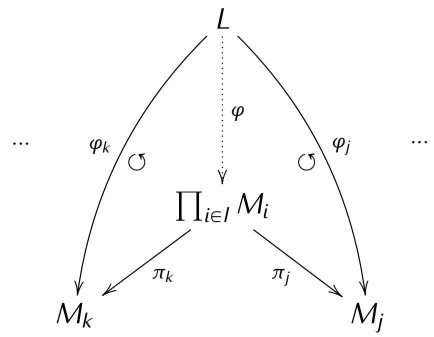
Let $\{M_i\}_{i\in I}$ be a family of $R$-modules. Then the abelian group $\prod_{i\in I} M_i$, that is the product of $\{M_i\}_{i\in I}$ seen as a family of abelian groups, becomes an $R$-module via the following external composition law:

$$\begin{aligned} R \times \prod_{i\in I} M_i &\longrightarrow \prod_{i\in I} M_i \\ \big(r, (m_i)_{i\in I}\big) &\longmapsto \big(r \cdot m_i\big)_{i\in I}. \end{aligned}$$

Furthermore, for each $j \in I$, we let $\pi_j : \prod_{i\in I} M_i \longrightarrow M_j, (m_i)_{i\in I} \mapsto m_j$ denotes the $j$-th projection from the product to the module $M_j$.

**Proposition C.1 (Universal property of the direct product)**

If $\{\varphi_i : L \longrightarrow M_i\}_{i\in I}$ is a family of $R$-homomorphisms, then there exists a unique $R$-homomorphism $\varphi : L \longrightarrow \prod_{i\in I} M_i$ such that $\pi_j \circ \varphi = \varphi_j$ for every $j \in I$.




Thus,

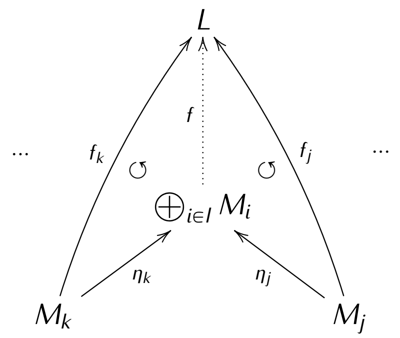
$$\begin{aligned} \mathrm{Hom}_R\Big(L, \prod_{i\in I} M_i\Big) &\longrightarrow \prod_{i\in I} \mathrm{Hom}_R(L, M_i) \\ f &\longmapsto \big(\pi_i \circ f\big)_{i\in I} \end{aligned}$$

is an isomorphism of abelian groups.

Now let $\bigoplus_{i\in I} M_i$ be the subgroup of $\prod_{i\in I} M_i$ consisting of the elements $(m_i)_{i\in I}$ such that $m_i = 0$ almost everywhere (i.e. $m_i = 0$ exept for a finite subset of indices $i \in I$). This subgroup is called the **direct sum** of the family $\{M_i\}_{i\in I}$ and is in fact an $R$-submodule of the product. For each $j \in I$, we let $\eta_j : M_j \longrightarrow \bigoplus_{i\in I} M_i, m_j \mapsto$ denote the canonical injection of $M_j$ in the direct sum.

**Proposition C.2 (Universal property of the direct sum)**

If $\{f_i : M_i \longrightarrow L\}_{i\in I}$ is a family of $R$-homomorphisms, then there exists a unique $R$-homomorphism $\varphi : \bigoplus_{i\in I} M_i \longrightarrow L$ such that $f \circ \eta_j = f_j$ for every $j \in I$.




Thus,

$$\begin{aligned} \mathrm{Hom}_R\Big(\bigoplus_{i\in I} M_i, L\Big) &\longrightarrow \prod_{i\in I} \mathrm{Hom}_R(M_i, L) \\ f &\longmapsto \big(f \circ \eta_i\big)_{i\in I} \end{aligned}$$

is an isomorphism of abelian groups.

**Remark C.3**

It is clear that if $|I| < \infty$, then $\bigoplus_{i\in I} M_i = \prod_{i\in I} M_i$.

The direct sum as defined above is often called an *external* direct sum. This relates as follows with the usual notion of *internal* direct sum:

**Remark C.4 (*"Internal" direct sums*)**

Let $M$ be an $R$-module and $N_1, N_2$ be two $R$-submodules of $M$. We write $M = N_1 \oplus N_2$ if every $m \in M$ can be written in a unique way as $m = n_1 + n_2$, where $n_1 \in N_1$ and $n_2 \in N_2$, or equivalently if $M = N_1 + N_2$ and $N_1 \cap N_2 = \{0\}$. In this case,

$$\begin{array}{rccl} \varphi : & M & \longrightarrow & N_1 \times N_2 = N_1 \oplus N_2 \ \text{(external direct sum)} \\ & m = n_1 + n_2 & \mapsto & (n_1, n_2)\,, \end{array}$$

is an isomorphism of $R$-modules.
This obviously generalises to arbitrary internal finite direct sums $M = \bigoplus_{i\in I} N_i$.

## D Exact sequences

Exact sequences constitute a very useful tool for the study of modules. Often we obtain valuable information about modules by *plugging them* in short exact sequences, where the other terms are known.

**Definition D.1 (*Exact sequence*)**

A sequence $L \xrightarrow{\varphi} M \xrightarrow{\psi} N$ of $R$-modules and $R$-linear maps is called **exact (at** $M$**)** if $\operatorname{Im}\varphi = \ker\psi$.

**Remark D.2 (*Injectivity/surjectivity/short exact sequences*)**

(a) $L \xrightarrow{\varphi} M$ is injective $\iff 0 \longrightarrow L \xrightarrow{\varphi} M$ is exact at $L$.

(b) $M \xrightarrow{\psi} N$ is surjective $\iff M \xrightarrow{\psi} N \longrightarrow 0$ is exact at $N$.

(c) $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ is exact (i.e. at $L$, $M$ and $N$) if and only if $\varphi$ is injective, $\psi$ is surjective and $\psi$ induces an $R$-isomorphism $\overline{\psi} : M/\operatorname{Im}\varphi \longrightarrow N, m + \operatorname{Im}\varphi \mapsto \psi(m)$.
Such a sequence is called a **short exact sequence (s.e.s.** for short).

(d) If $\varphi \in \operatorname{Hom}_R(L, M)$ is an injective morphism, then there is a s.e.s.

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\pi} \operatorname{coker}(\varphi) \longrightarrow 0$$

where $\pi$ is the canonical projection.

(e) If $\psi \in \operatorname{Hom}_R(M, N)$ is a surjective morphism, then there is a s.e.s.

$$0 \longrightarrow \ker(\psi) \xrightarrow{i} M \xrightarrow{\psi} N \longrightarrow 0\,,$$

where $i$ is the canonical injection.

**Proposition D.3**

Let $Q$ be an $R$-module. Then the following holds:

(a) $\operatorname{Hom}_R(Q, -) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a *left* exact covariant functor. In other words, if $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ is a s.e.s of $R$-modules, then the induced sequence

$$0 \longrightarrow \operatorname{Hom}_R(Q, L) \xrightarrow{\varphi_*} \operatorname{Hom}_R(Q, M) \xrightarrow{\psi_*} \operatorname{Hom}_R(Q, N)$$

is an exact sequence of abelian groups. Here $\varphi_* := \operatorname{Hom}_R(Q, \varphi)$, that is $\varphi_*(\alpha) = \varphi \circ \alpha$ for every $\alpha \in \operatorname{Hom}_R(Q, L)$ and similarly for $\psi_*$.

(b) $\operatorname{Hom}_R(-, Q) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a *left* exact contravariant functor. In other words, if $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ is a s.e.s of $R$-modules, then the induced sequence

$$0 \longrightarrow \operatorname{Hom}_R(N, Q) \xrightarrow{\psi^*} \operatorname{Hom}_R(M, Q) \xrightarrow{\varphi^*} \operatorname{Hom}_R(L, Q)$$

is an exact sequence of abelian groups. Here $\varphi^* := \operatorname{Hom}_R(\varphi, Q)$, that is $\varphi^*(\alpha) = \alpha \circ \varphi$ for every $\alpha \in \operatorname{Hom}_R(M, Q)$ and similarly for $\psi^*$.

**Remark D.4**

Notice that $\operatorname{Hom}_R(Q, -)$ and $\operatorname{Hom}_R(-, Q)$ are not *right* exact in general.

**Lemma-Definition D.5 (*Split short exact sequence*)**

A s.e.s. $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ of $R$-modules is called **split** if it satisfies one of the following equivalent conditions:

(a) $\psi$ admits an $R$-linear section, i.e. if $\exists\, \sigma \in \mathrm{Hom}_R(N, M)$ such that $\psi \circ \sigma = \mathrm{Id}_N$;

(b) $\varphi$ admits an $R$-linear retraction, i.e. if $\exists\ \rho \in \mathrm{Hom}_R(M, L)$ such that $\rho \circ \varphi = \mathrm{Id}_L$;

(c) $\exists$ an $R$-isomorphism $\alpha : M \longrightarrow L \oplus N$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}_L} & \circlearrowleft & \downarrow{\scriptstyle \alpha} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{Id}_N} & & \\ 0 & \longrightarrow & L & \xrightarrow{i} & L \oplus N & \xrightarrow{p} & N & \longrightarrow & 0\,, \end{array}$$

where $i$, resp. $p$, are the canonical inclusion, resp. projection.

**Remark D.6**

If the sequence splits and $\sigma$ is a section, then $M = \varphi(L) \oplus \sigma(N)$. If the sequence splits and $\rho$ is a retraction, then $M = \varphi(L) \oplus \ker(\rho)$.

**Example D.7**

The s.e.s. of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

defined by $\varphi([1]) = ([1], [0])$ and where $\pi$ is the canonical projection onto the cokernel of $\varphi$ is split but the sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

defined by $\varphi([1]) = ([2])$ and $\pi$ is the canonical projection onto the cokernel of $\varphi$ is not split.

# E Tensor products

**Definition E.1 (*Tensor product of R-modules*)**

Let $M$ be a right $R$-module and let $N$ be a left $R$-module. Let $F$ be the free $\mathbb{Z}$-module with basis $M \times N$. Let $G$ be the subgroup of $F$ generated by all the elements

$$\begin{aligned} &(m_1 + m_2, n) - (m_1, n) - (m_2, n), \quad \forall m_1, m_2 \in M, \forall n \in N, \\ &(m, n_1 + n_2) - (m, n_1) - (m, n_2), \quad \forall m \in M, \forall n_1, n_2 \in N, \text{ and} \\ &(mr, n) - (m, rn), \quad \forall m \in M, \forall n \in N, \forall r \in R. \end{aligned}$$

The **tensor product of $M$ and $N$ (balanced over $R$)**, is the abelian group $M \otimes_R N := F/G$. The class of $(m, n) \in F$ in $M \otimes_R N$ is denoted by $m \otimes n$.

**Remark E.2**

(a) $M \otimes_R N = \langle m \otimes n \mid m \in M, n \in N \rangle_{\mathbb{Z}}$.

(b) In $M \otimes_R N$, we have the relations

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \quad \forall m_1, m_2 \in M, \forall n \in N,$$
$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \quad \forall m \in M, \forall n_1, n_2 \in N, \text{ and}$$
$$mr \otimes n = m \otimes rn, \quad \forall m \in M, \forall n \in N, \forall r \in R.$$

In particular, $m \otimes 0 = 0 = 0 \otimes n \ \forall \ m \in M, \forall \ n \in N$ and $(-m) \otimes n = -(m \otimes n) = m \otimes (-n)$ $\forall \ m \in M, \forall \ n \in N$.

**Definition E.3 (*R-balanced map*)**

Let $M$ and $N$ be as above and let $A$ be an abelian group. A map $f : M \times N \longrightarrow A$ is called **$R$-balanced** if

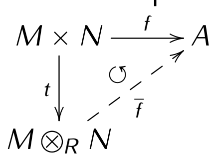
$$f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n), \quad \forall m_1, m_2 \in M, \forall n \in N,$$
$$f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2), \quad \forall m \in M, \forall n_1, n_2 \in N,$$
$$f(mr, n) = f(m, rn), \quad \forall m \in M, \forall n \in N, \forall r \in R.$$

**Remark E.4**

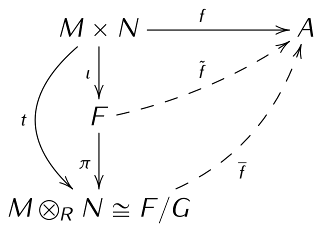
The canonical map $t : M \times N \longrightarrow M \otimes_R N, (m, n) \mapsto m \otimes n$ is $R$-balanced.

**Proposition E.5 (*Universal property of the tensor product*)**

Let $M$ be a right $R$-module and let $N$ be a left $R$-module. For every abelian group $A$ and every $R$-balanced map $f : M \times N \longrightarrow A$ there exists a unique $\mathbb{Z}$-linear map $\bar{f} : M \otimes_R N \longrightarrow A$ such that the following diagram commutes:




**Proof:** Let $\iota : M \times N \longrightarrow F$ denote the canonical inclusion, and let $\pi : F \longrightarrow F/G$ denote the canonical projection. By the universal property of the free $\mathbb{Z}$-module, there exists a unique $\mathbb{Z}$-linear map $\tilde{f} : F \longrightarrow A$ such that $\tilde{f} \circ \iota = f$. Since $f$ is $R$-balanced, we have that $G \subseteq \ker(\tilde{f})$. Therefore, the universal property of the quotient yields the existence of a unique homomorphism of abelian groups $\bar{f} : F/G \longrightarrow A$ such that $\bar{f} \circ \pi = \tilde{f}$:




Clearly $t = \pi \circ \iota$, and hence $\bar{f} \circ t = \bar{f} \circ \pi \circ \iota = \tilde{f} \circ \iota = f$. ■

**Remark E.6**

Let $M$ and $N$ be as in Definition E.1.

(a) Let $\{M_i\}_{i\in I}$ be a collection of right $R$-modules, $M$ be a right $R$-module, $N$ be a left $R$-module and $\{N_j\}_{i\in J}$ be a collection of left $R$-modules. Then, we have

$$\bigoplus_{i\in I} M_i \otimes_R N \cong \bigoplus_{i\in I}(M_i \otimes_R N)$$
$$M \otimes_R \bigoplus_{j\in J} N_j \cong \bigoplus_{j\in J}(M \otimes_R N_j).$$

(This is easily proved using both the universal property of the direct sum and of the tensor product.)

(b) There are natural isomorphisms of abelian groups given by $R \otimes_R N \cong N$ via $r \otimes n \mapsto rn$, and $M \otimes_R R \cong M$ via $m \otimes r \mapsto mr$.

(c) It follows from (b), that if $P$ is a free left $R$-module with $R$-basis $X$, then $N \otimes_R P \cong \bigoplus_{x\in X} N$, and if $P$ is a free right $R$-module with $R$-basis $X$, then $P \otimes_R M \cong \bigoplus_{x\in X} M$.

(d) Let $Q$ be a third ring. Then we obtain module structures on the tensor product as follows:

(i) If $M$ is a $(Q, R)$-bimodule and $N$ a left $R$-module, then $M \otimes_R N$ can be endowed with the structure of a left $Q$-module via

$$q \cdot (m \otimes n) = q \cdot m \otimes n \quad \forall q \in Q, \forall m \in M, \forall n \in N.$$

(ii) If $M$ is a right $R$-module and $N$ an $(R, S)$-bimodule, then $M \otimes_R N$ can be endowed with the structure of a right $S$-module via

$$(m \otimes n) \cdot s = qm \otimes n \cdot s \quad \forall s \in S, \forall m \in M, \forall n \in N.$$

(iii) If $M$ is a $(Q, R)$-bimodule and $N$ an $(R, S)$-bimodule. Then $M \otimes_R N$ can be endowed with the structure of a $(Q, S)$-bimodule via the external composition laws defined in (i) and (ii).

(e) Assume $R$ is commutative. Then any $R$-module can be viewed as an $(R, R)$-bimodule. Then, in particular, $M \otimes_R N$ becomes an $R$-module (both on the left and on the right).

(f) For instance, it follows from (e) that if $K$ is a field and $M$ and $N$ are $K$-vector spaces with $K$-bases $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ resp., then $M \otimes_K N$ is a $K$-vector space with a $K$-basis given by $\{x_i \otimes y_j\}_{(i,j)\in I\times J}$.

(g) **Tensor product of morphisms:** Let $f : M \longrightarrow M'$ be a morphism of right $R$-modules and $g : N \longrightarrow N'$ be a morphism of left $R$-modules. Then, by the universal property of the tensor product, there exists a unique $\mathbb{Z}$-linear map $f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$ such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.

**Proposition E.7 (*Right exactness of the tensor product*)**

(a) Let $N$ be a left $R$-module. Then $-\otimes_R N : \textbf{Mod}_R \longrightarrow \textbf{Ab}$ is a right exact covariant functor.

(b) Let $M$ be a right $R$-module. Then $M \otimes_R - : {}_R\textbf{Mod} \longrightarrow \textbf{Ab}$ is a right exact covariant functor.

**Remark E.8**

The functors $-\otimes_R N$ and $M \otimes_R -$ are not left exact in general.

# F Algebras

**Definition F.1 (*Algebra*)**

Let $R$ be a commutative ring.

(a) An $R$**-algebra** is an ordered quadruple $(A, +, \cdot, *)$ such that the following axioms hold:

**(A1)** $(A, +, \cdot)$ is a ring;
**(A2)** $(A, +, *)$ is a left $R$-module; and
**(A3)** $r * (a \cdot b) = (r * a) \cdot b = a \cdot (r * b)\ \forall\ a, b \in A,\ \forall\ r \in R$.

(b) A map $f : A \to B$ between two $R$-algebras is called an **algebra homomorphism** iff:

(i) $f$ is a homomorphism of $R$-modules;
(ii) $f$ is a ring homomorphism.

**Example F.2 (*Algebras*)**

(a) The commutative ring $R$ itself is an $R$-algebra.
[The internal composition law "$\cdot$" and the external composition law "$*$" coincide in this case.]

(b) For each $n \in \mathbb{Z}_{\geqslant 1}$ the set $M_n(R)$ of $n \times n$-matrices with coefficients in $R$ is an $R$-algebra for its usual $R$-module and ring structures.
[Note: in particular $R$-algebras need not be commutative rings in general!]

(c) Let $K$ be a field. Then for each $n \in \mathbb{Z}_{\geqslant 1}$ the polynom ring $K[X_1, \ldots, X_n]$ is a $K$-algebra for its usual $K$-vector space and ring structure.

(d) $\mathbb{R}$ and $\mathbb{C}$ are $\mathbb{Q}$-algebras, $\mathbb{C}$ is an $\mathbb{R}$-algebra, ...

(e) Rings are $\mathbb{Z}$-algebras.

**Example F.3 (*Modules over algebras*)**

(a) $A = M_n(R) \Rightarrow R^n$ is an $A$-module for the external composition law given by left matrix multiplication $A \times R^n \longrightarrow R^n, (B, x) \mapsto Bx$.

(b) If $K$ is a field and $V$ a $K$-vector space, then $V$ becomes an $A$-algebra for $A := \mathrm{End}_K(V)$ together with the external composition law

$$A \times V \longrightarrow V, (\varphi, v) \mapsto \varphi(v).$$

(c) An arbitrary $A$-module $M$ can be seen as an $R$-module via a change of the base ring since $R \longrightarrow A, r \mapsto r * 1_A$ is a homomorphism of rings by the algebra axioms.

**Remark F.4**

Let $R$ be a commutative ring.

(a) Let $M, N$ be $R$-modules. Prove that:

(1) $\mathrm{End}_R(M)$, endowed with the pointwise addition of maps and the usual composition of maps, is a ring. (Note that the commutativity of $R$ is not necessary!)

(2) The abelian group $\mathrm{Hom}_R(M, N)$ is a left $R$-module for the external composition law defined by

$$(rf)(m) := f(rm) = rf(m) \qquad \forall\, r \in R, \forall f \in \mathrm{Hom}_R(M, N), \forall m \in M.$$

It follows that $\mathrm{End}_R(M)$ is an $R$-algebra.

(b) Let now $A$ be an $R$-algebra and $M$ be an $A$-module. Then $\mathrm{End}_R(M)$ and $\mathrm{End}_A(M)$ are $R$-algebras.