(Ir)Reducibility, (In)Decomposability, Semisimplicity

Notation. Below R denotes an arbitrary unital and associative ring.

K (Ir)Reducibility and (in)decomposability

Submodules and direct sums of modules lead to two main notions that enable us to break modules in *elementary* pieces in order to simplify their study: *simplicity* and *indecomposability*.

Definition K.1 (*simple/irreducible module / indecomposable module / semisimple module*)

(a) An R-module M is called **reducible** if it admits an R-submodule U such that $0 \subsetneq U \subsetneq M$. An R-module M is called **simple**, or **irreducible**, if it is non-zero and not reducible.

(b) An R-module M is called **decomposable** if M possesses two non-zero proper submodules M_1, M_2 such that $M = M_1 \oplus M_2$. An R-module M is called **indecomposable** if it is non-zero and not decomposable.

(c) An R-module M is called **completely reducible** or **semisimple** if it admits a direct sum decomposition into simple R-submodules.

When R is the group algebra of a finite group, we will investigate each of these three concepts in details in the lectures.

Remark K.2

Clearly any simple module is also indecomposable, resp. semisimple. However, the converse does not hold in general.

Remark K.3

If $(R, +, \cdot)$ is a ring, then $R^\circ := R$ itself maybe seen as an R-module, called the **regular** module, where the external composition law is given by left multiplication, i.e.

$$R \times R^\circ \longrightarrow R^\circ, (r, m) \mapsto r \cdot m\,.$$

Ideals and submodules may be compared as follows:

(a) the R-submodules of R° are precisely the left ideals of R;

(b) $I \lhd R$ is a maximal left ideal of $R \Leftrightarrow R^{\circ}/I$ is a simple R-module, and $I \lhd R$ is a minimal left ideal of $R \Leftrightarrow I$ is simple when regarded as an R-submodule of R°.

L Semisimplicity of rings and modules

There are several equivalent characterisations of semisimplicity. We need the following ones.

Proposition L.1

If M is an R-module, then the following assertions are equivalent:

(a) M is semisimple, i.e. $M = \oplus_{i\in I} S_i$ for some family $\{S_i\}_{i\in I}$ of simple R-submodules of M;

(b) $M = \sum_{i\in I} S_i$ for some family $\{S_i\}_{i\in I}$ of simple R-submodules of M;

(c) every R-submodule $M_1 \subseteq M$ admits a complement in M, i.e. $\exists$ an R-submodule $M_2 \subseteq M$ such that $M = M_1 \oplus M_2$.

Example 1

(a) The zero module is completely reducible, but neither reducible nor irreducible!

(b) If $S_1, \ldots, S_n$ are simple R-modules, then their direct sum $S_1 \oplus \ldots \oplus S_n$ is completely reducible by definition.

(c) The following exercise shows that there exists modules which are not completely reducible.

Exercise: Let K be a field and let A be the K-algebra $\left\{\left(\begin{smallmatrix} a_1 & a \\ 0 & a_1 \end{smallmatrix}\right) \mid a_1, a \in K\right\}$. Consider the A-module $V := K^2$, where A acts by left matrix multiplication. Prove that:

(1) $\{\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \mid x \in K\}$ is a simple A-submodule of V; but
(2) V is not semisimple.

(d) Any submodule and any quotient of a completely reducible module is again completely reducible.

Theorem-Definition L.2 (*Semisimple ring*)

A ring R satisfying the following equivalent conditions is called **semisimple**.

(a) All short exact sequences of R-modules split.

(b) All R-modules are semisimple.

(c) All finitely generated R-modules are semisimple.

(d) The regular left R-module R° is semisimple, and is a direct sum of a finite number of minimal left ideals.

Example 2

Fields are semisimple. Indeed, if V is a finite-dimensional vector space over a field K of dimension n, then choosing a K-basis $\{e_1, \cdots, e_n\}$ of V yields $V = Ke_1 \oplus \ldots \oplus Ke_n$, where $\dim_K(Ke_i) = 1$, hence Ke_i is a simple K-module for each $1 \leqslant i \leqslant n$.

Corollary L.3

Let R be a semisimple ring. Then:

(a) R° has a composition series;

(b) R is both left Artinian and left Noetherian.

Next, we show that semisimplicity is detected by the Jacobson radical. This leads us to introduce a slightly weaker concept: the notion of *J-semisimplicity*.

Definition L.4 (*J-semimplicity*)

A ring R is said to be **J-semisimple** if $J(R) = 0$.

Remark L.5

The ring of integers $\mathbb{Z}$ is J-semisimple but not semisimple, because $J(\mathbb{Z}) = 0$, but not all $\mathbb{Z}$-modules are semisimple.

However:

Proposition L.6

Any left Artinian ring R is J-semisimple if and only if it is semisimple.

Proposition L.7

The quotient ring $R/J(R)$ is J-semisimple.

Proof: Since the rings R and $\overline{R} := R/J(R)$ have the same simple modules (seen as abelian groups), Proposition-Definition H.1(a) yields:

$$J(\overline{R}) = \bigcap_{\substack{V \text{simple} \\ \overline{R}\text{-module}}} \operatorname{ann}_{\overline{R}}(V) = \bigcap_{\substack{V \text{simple} \\ R\text{-module}}} \operatorname{ann}_R(V) + J(R) = J(R)/J(R) = 0$$

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