Throughout, unless otherwise stated, K denotes a field of arbitrary characteristic, G a finite group and all K-vector spaces are finite-dimensional. Each Exercise is worth 4 points.

Exercise 1

(a) Let $G := S_3 = \langle (1\,2), (1\,2\,3)\rangle$ and $K = \mathbb{C}$. Prove that

$$\rho_1 : S_3 \longrightarrow \mathbb{C}^\times, \sigma \mapsto 1\,,$$
$$\rho_2 : S_3 \longrightarrow \mathbb{C}^\times, \sigma \mapsto \mathrm{sign}(\sigma)\,,$$
$$\begin{array}{cccc} \rho_3 : & S_3 & \longrightarrow & \mathrm{GL}_2(\mathbb{C}) \\ & (1\,2) & \mapsto & \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \\ & (1\,2\,3) & \mapsto & \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right) \end{array}$$

are three non-equivalent irreducible matrix representations of G.

(b) Assume $char(K) \neq 2$ and let $G := C_2 \times C_2$ be the Klein-four group. Find four pairwise non-equivalent (matrix) representations of G over K of degree one.

Exercise 2

(a) Prove that the trivial representation of G is a subrepresentation of any permutation representation of G over K.

(b) Assume $K = \mathbb{C}$ and $G = C_3$. Find a decomposition into a direct sum of three irreducible subrepresentations of the regular representation of G.
[Hint: use (a) and $\omega \in \mathbb{C}$ a cubic root of unity to find three G-invariant subspaces of $V := \langle \{e_g \mid g \in G\}\rangle_K$]

Exercise 3

Let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two K-representations of G and let $\alpha : V_1 \longrightarrow V_2$ be a G-homomorphism. Prove the following assertions.

(a) If $W \subseteq V_1$ is a G-invariant subspace of V_1, then $\alpha(W) \subseteq V_2$ is G-invariant.

(b) If $W \subseteq V_2$ is a G-invariant subspace of V_2, then $\alpha^{-1}(W) \subseteq V_1$ is G-invariant.

(c) Both $\ker(\alpha)$ and $\mathrm{Im}(\alpha)$ are G-invariant subspaces of V_1 and V_2 respectively.

Exercise 4 (Maschke's Theorem does not hold without the assumption that $char(K) \nmid |G|$.)
Let p be a prime number, let $G := C_p = \langle g \mid g^p = 1\rangle$ and let $K := \mathbb{F}_p$. Let $B := (e_1, e_2)$ be the standard ordered basis of $V := K^2$. Consider the matrix representation

$$\begin{array}{cccc} R : & G & \longrightarrow & GL_2(K) \\ & g^b & \mapsto & \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right). \end{array}$$

(a) Prove that $W := Ke_1$ is G-invariant and deduce that R is reducible.

(b) Prove that there is no direct sum decomposition of V into irreducible G-invariant subspaces.