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Due date: **Thursday the 7th of May, 2020, 13:00 in OpenOLAT** SS 2020

Throughout this exercise sheet K denotes a field of arbitrary characteristic, $(G,\cdot)$ a finite group with neutral element 1_G, V a finite-dimensional K-vector space.
Each Exercise is worth 4 points.

Exercise 5 (Alternative proof of Maschke's Theorem over the field $\mathbb{C}$.)
Assume $K=\mathbb{C}$ and let $\rho: G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G.

(a) Prove that there exists a G-invariant scalar product $\langle\ ,\ \rangle : V\times V \longrightarrow \mathbb{C}$, i.e. such that

$$\langle g.u, g.v\rangle = \langle u,v\rangle \quad \forall\, g\in G, \forall\, u,v\in V\,.$$

[Hint: consider an arbitrary scalar product on V, say $(\ ,\ ) : V\times V \longrightarrow \mathbb{C}$, which is not necessarily G-invariant. Use a sum on the elements of G weighted by the group order in order to produce a new G-invariant scalar product on V.]

(b) Deduce that every G-invariant subspace W of V admits a G-invariant complement. [Hint: consider the orthogonal complement of W.]

Exercise 6
Assume we are in the situation of Proposition 4.3. Namely, we are given a K-vector space $(V,+,\cdot)$ and we define an external multiplication on V by the elements of KG through a left action $G\times V \longrightarrow V, (g,v)\mapsto g\cdot v$ of G on V which we extend by K-linearity to the whole of KG. Thus, we now have a KG-module $(V,+,\cdot)$, where the new external multiplication $\cdot : KG \longrightarrow V$ extends the initial external multiplication on V by the elements of K.
Prove that checking the KG-module axioms (Appendix A, Definition A.1) for $(V,+,\cdot)$ is equivalent to checking the following axioms:

(1) $(gh)\cdot v = g\cdot(h\cdot v)\,,$

(2) $1_G\cdot v = v\,,$

(4) $g\cdot(u+v) = g\cdot u + g\cdot v\,,$

(3) $g\cdot(\lambda v) = \lambda(g\cdot v) = (\lambda g)\cdot v\,,$

for all $g,h\in G$, $\lambda\in K$ and $u,v\in V$.

Exercise 7

(a) Check the details of the proof of Proposition 4.3.
[Hint: use Exercise 6.]

(b) Use Proposition 4.3 to express the trivial representation in terms of KG-modules.

(c) Use Proposition 4.3 to express the regular representation in terms of KG-modules. Prove that the KG-module you have obtained is isomorphic to KG (the group algebra) seen as left KG-module over itself.

Exercise 8 (**Schur's Lemma for matrix representations**)
Let $R : G \longrightarrow \mathrm{GL}_n(K)$ and $R' : G \longrightarrow \mathrm{GL}_{n'}(K)$ be two irreducible matrix representations. Prove that if there exists $A \in M_{n\times n'}(K)\setminus\{0\}$ such that $AR'(g) = R(g)A$ for every $g \in G$, then $n = n'$ and A is invertible (in particular $R \sim R'$).