Throughout this exercise sheet $K = \mathbb{C}$ is the field of complex numbers, $(G, \cdot)$ is a finite group, and V a finite-dimensional $\mathbb{C}$-vector space. Each Exercise is worth 4 points.

Exercise 17

(a) Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G with character χ. Prove that

$$\ker(\chi) = \ker(\rho)\,,$$

thus is a normal subgroup of G.

(b) Prove that if $N \trianglelefteq G$, then

$$N = \bigcap_{\substack{\chi \in \mathrm{Irr}(G) \\ N \subseteq \ker(\chi)}} \ker(\chi)\,.$$

Exercise 18

Prove that a finite group G is simple if and only if $\chi(g) \neq \chi(1)$ for each $g \in G \setminus \{1\}$ and each $\chi \in \mathrm{Irr}(G) \setminus \{\mathbf{1}_G\}$.

Exercise 19

Compute the complex character table of the alternating group A_4 through the following steps:

1. Determine the conjugacy classes of A_4 (there are 4 of them) and the corresponding centraliser orders. [Justify all your computations.]
2. Determine the degrees of the 4 irreducible characters of A_4.
3. Determine the linear characters of A_4.
4. Determine the non-linear character of A_4 using the 2nd Orthogonality Relations.

Exercise 20

(a) Compute the character tables of the dihedral group D_8 of order 8 and of the quaternion group Q_8.

[Hint: In each case, determine the commutator subgroup and deduce that there are 4 linear characters.]

(b) If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of G and $\det : \mathrm{GL}(V) \longrightarrow \mathbb{C}^*$ denotes the determinant homomorphism, then we define a linear character of G through

$$\det_\rho := \det \circ \rho : G \longrightarrow \mathbb{C}^*\,,$$

called the **determinant of** ρ. Prove that, although the finite groups D_8 and Q_8 have the "same" character table, they can be distinguished by considering the determinants of their irreducible $\mathbb{C}$-representations.