Throughout this exercise sheet $K = \mathbb{C}$ is the field of complex numbers, $(G, \cdot)$ is a finite group, and V a finite-dimensional $\mathbb{C}$-vector space. Each Exercise is worth 4 points.

Exercise 21
Prove the following assertions:

(a) If G is a non-abelian simple group (or more generally if G is perfect, i.e. $G = [G, G]$), then the image $\rho(G)$ of any representation $\rho : G \longrightarrow \mathrm{GL}(V)$ is a subgroup of $\mathrm{SL}(V)$.

(b) No simple group G has an irreducible character of degree 2.
Assume that G is simple and $\rho : G \longrightarrow \mathrm{GL}_2(\mathbb{C})$ is an irreducible matrix representation of G with character χ and proceed as follows:

1. Prove that ρ is faithful and G is non-abelian.
2. Determine the determinant $\det_\rho$ of ρ.
3. Prove that $|G|$ is even and G admits an element x of order 2.
4. Prove that $\langle x\rangle \lhd G$ and conclude that assertion (b) holds.
(Use the diagonalisation theorem and steps 1., 2. and 3.)

Exercise 22
Let G be a finite group of odd order and, as usual, let r denote the number of conjugacy classes of G. Use character theory to prove that

$$r \equiv |G| \pmod{16}.$$

[Hint: Label the set $\mathrm{Irr}(G)$ of irreducible characters taking dual characters into account.]

Exercise 23
Prove that if $\chi \in \mathrm{Irr}(G)$, then $Z(G) \leq Z(\chi)$ and deduce that $\bigcap_{\chi\in\mathrm{Irr}(G)} Z(\chi) = Z(G)$.

Exercise 24

(a) Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be an irreducible faithful $\mathbb{C}$-representation with character χ. Let $m \in \mathbb{Z}_{\geq 1}$ and let $\rho^{\otimes m} := \rho \otimes \cdots \otimes \rho : G^m \longrightarrow \mathrm{GL}(V^{\otimes m})$ be the m-fold tensor product of ρ with itself.

(i) Prove that $Z(\chi) = Z(G)$;

(ii) Prove that $H := \{(z_1, \ldots, z_m) \in Z(G^m) \mid z_1 \cdots z_m = 1\}$ is a normal subgroup of G^m such that $|H| = |Z(G)|^{m-1}$ and $H \leq \ker(\rho^{\otimes m})$.

(iii) Prove that $\rho^{\otimes m}$ induces an irreducible $\mathbb{C}$-representation of G^m/H and deduce that $\chi(1)^m \mid \frac{|G|^m}{|Z(G)|^{m-1}}$.

(iv) Set $\alpha := \frac{\chi(1)}{\gcd(\chi(1), |G:Z(G)|)}$ and prove that $\alpha^m \leq |Z(G)|$.

(v) Use (iv) to determine α and deduce that $\chi(1) \mid |G : Z(G)|$.

(b) Deduce from (a) that $\chi(1) \mid |G : Z(\chi)|$ for every irreducible character $\chi \in \mathrm{Irr}(G)$.
(Hint: mod out by the kernel of χ.)