Throughout this exercise sheet $K = \mathbb{C}$ is the field of complex numbers, $(G, \cdot)$ is a finite group, and V a finite-dimensional $\mathbb{C}$-vector space. Each Exercise is worth 4 bonus points.

Exercise 25
Let $H \leq J \leq G$. Prove the following assertions:

(a) $\varphi \in Cl(H) \implies (\varphi\uparrow_H^J)\uparrow_J^G = \varphi\uparrow_H^G$ (transitivity of induction);

(b) $\psi \in Cl(G) \implies (\psi\downarrow_J^G)\downarrow_H^J = \psi\downarrow_H^G$ (transitivity of restriction);

(c) $\varphi \in Cl(H)$ and $\psi \in Cl(G) \implies \psi \cdot \varphi\uparrow_H^G = \left(\psi\downarrow_H^G \cdot \varphi\right)\uparrow_H^G$ (Frobenius formula);

(d) the map $\mathrm{Ind}_H^G : Cl(H) \longrightarrow Cl(G), \varphi \mapsto \varphi\uparrow_H^G$ is $\mathbb{C}$-linear.

Exercise 26
With the notation of Definition 20.1, prove that:

(a) ${}^g\varphi$ is indeed a class function on gHg^{-1};

(b) $\mathcal{I}_G(\varphi) \leq G$ and $H \leq \mathcal{I}_G(\varphi) \leq N_G(H)$;

(c) for $g, h \in G$ we have ${}^g\varphi = {}^h\varphi \iff h^{-1}g \in \mathcal{I}_G(\varphi) \iff g\mathcal{I}_G(\varphi) = h\mathcal{I}_G(\varphi)$;

(d) if $\rho : H \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of H with character χ, then

$$ {}^g\rho : gHg^{-1} \longrightarrow \mathrm{GL}(V), x \mapsto \rho(g^{-1}xg) $$

is $\mathbb{C}$-representation of gHg^{-1} with character ${}^g\chi$ and ${}^g\chi(1) = \chi(1)$;

(e) if $J \leq H$ then ${}^g(\varphi\downarrow_J^H) = ({}^g\varphi)\downarrow_{gJg^{-1}}^{gHg^{-1}}$.

Exercise 27
Let $A \leq G$ be an abelian subgroup of G and let $\chi \in \mathrm{Irr}(G)$. Prove that $\chi(1) \leq |G : A|$.

Exercise 28
Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Prove that

$$ \chi\downarrow_N^G\uparrow_N^G = \mathrm{Inf}_{G/N}^G(\chi_{\mathrm{reg}}) \cdot \chi\,, $$

where χ_{reg} is the regular character of G/N.