

Throughout, unless otherwise stated, K denotes a field of arbitrary characteristic, G a finite group and all K -vector spaces are finite-dimensional.

EXERCISE 1

- (a) Let $G := S_3 = \langle (1\ 2), (1\ 2\ 3) \rangle$ and $K = \mathbb{C}$. Prove that

$$\begin{aligned} \rho_1 : S_3 &\longrightarrow \mathbb{C}^\times, \sigma \mapsto 1, \\ \rho_2 : S_3 &\longrightarrow \mathbb{C}^\times, \sigma \mapsto \text{sign}(\sigma), \\ \rho_3 : \quad S_3 &\longrightarrow \text{GL}_2(\mathbb{C}) \\ (1\ 2) &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ (1\ 2\ 3) &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

are three non-equivalent irreducible matrix representations of G .

- (b) Assume $\text{char}(K) \neq 2$ and let $G := C_2 \times C_2$ be the Klein-four group. Find four pairwise non-equivalent (matrix) representations of G over K of degree one.

EXERCISE 2

Prove that the trivial representation of G is a subrepresentation of any permutation representation of G over K .

EXERCISE 3 (Exercise to hand in / 8 points)

- (i) Let $\rho_1 : G \rightarrow \text{GL}(V_1)$ and $\rho_2 : G \rightarrow \text{GL}(V_2)$ be two K -representations of G and let $\alpha : V_1 \rightarrow V_2$ be a G -homomorphism. Prove the following assertions.
- If $W \subseteq V_1$ is a G -invariant subspace of V_1 , then $\alpha(W) \subseteq V_2$ is G -invariant.
 - If $W \subseteq V_2$ is a G -invariant subspace of V_2 , then $\alpha^{-1}(W) \subseteq V_1$ is G -invariant.
 - Both $\ker(\alpha)$ and $\text{Im}(\alpha)$ are G -invariant subspaces of V_1 and V_2 respectively.
- (ii) Assume $K = \mathbb{C}$ and $G = C_3$. Find a decomposition into a direct sum of three irreducible subrepresentations of the regular representation of G .

EXERCISE 4 (Maschke's Theorem does not hold without the assumption that $\text{char}(K) \nmid |G|$.)

Let p be a prime number, let $G := C_p = \langle g \mid g^p = 1 \rangle$ and let $K := \mathbb{F}_p$. Let $B := (e_1, e_2)$ be the standard ordered basis of $V := K^2$. Consider the matrix representation

$$\begin{aligned} R : \quad G &\longrightarrow \text{GL}_2(K) \\ g^b &\mapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

- Prove that $W := Ke_1$ is G -invariant and deduce that R is reducible.
- Prove that there is no direct sum decomposition of V into irreducible G -invariant subspaces.