C Tensor Products of Vector Spaces

Throughout this section, we assume that K is a field.

Definition C.1 (*Tensor product of vector spaces*)

Let V, W be two finite-dimensional K-vector spaces with bases $B_V = \{v_1, \dots, v_n\}$ and $B_W = \{w_1, \dots, w_m\}$ ($m, n \in \mathbb{Z}_{\geqslant 0}$) respectively. The **tensor product of V and W (balanced) over K** is by definition the $(n \cdot m)$-dimensional K-vector space

$$V \otimes_K W$$

with basis $B_{V \otimes_K W} = \{v_i \otimes w_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$.

In this definition, you should understand the symbole "$v_i \otimes w_j$" as an element that depends on both v_i and w_j. The symbole "$\otimes$" itself does not have any hidden meaning, it is simply a piece of notation: we may as well write something like $x(v_i, w_j)$ instead of "$v_i \otimes w_j$", but we have chosen to write "$v_i \otimes w_j$".

Properties C.2

(a) An arbitrary element of $V \otimes_K W$ has the form

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij}(v_i \otimes w_j) \quad \text{with } \{\lambda_{ij}\}_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} \subseteq K\,.$$

(b) The binary operation

$$\begin{array}{ccc} B_V \times B_W & \longrightarrow & B_{V \otimes_K W} \\ (v_i, w_j) & \mapsto & v_i \otimes w_j \end{array}$$

can be extended by $\mathbb{C}$-linearity to

$$\begin{array}{cccc} - \otimes - : & V \times W & \longrightarrow & V \otimes_K W \\ & \big(v = \sum_{i=1}^{n} \lambda_i v_i, w = \sum_{i=1}^{n} \mu_j w_j\big) & \mapsto & v \otimes w = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j (v_i \otimes w_j)\,. \end{array}$$

It follows that $\forall\ v \in V, w \in W, \lambda \in K$,

$$v \otimes (\lambda w) = (\lambda v) \otimes w = \lambda(v \otimes w)\,,$$

and $\forall\ x_1, \dots, x_r \in V, y_1, \dots y_s \in W$,

$$\Big(\sum_{i=1}^{r} x_i\Big) \otimes \Big(\sum_{j=1}^{s} y_j\Big) = \sum_{i=1}^{r} \sum_{j=1}^{s} x_i \otimes y_j\,.$$

Thus any element of $V \otimes_K W$ may also be written as a K-linear combination of elements of the form $v \otimes w$ with $v \in V, w \in W$. In other words $\{v \otimes w \mid v \in V, w \in W\}$ generates $V \otimes_K W$ (although it is not a K-basis).

(c) Up to isomorphism $V \otimes_K W$ is independent of the choice of the K-bases of V and W.

Definition C.3 (*Kronecker product*)

If $A = \big(A_{ij}\big)_{ij} \in M_n(K)$ and $B = \big(B_{rs}\big)_{rs} \in M_m(K)$ are two square matrices, then their **Kronecker product** (or **tensor product**) is the matrix

$$A \otimes B = \begin{bmatrix} A_{11}B \cdots\cdots A_{1n}B \\ \vdots \qquad\qquad \vdots \\ A_{n1}B \cdots\cdots A_{nn}B \end{bmatrix} \in M_{n\cdot m}(K)$$

Notice that it is clear from the above definition that $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$.

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E.g. the tensor product of two 2×2-matrices is of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix} \in M_4(K)\,.$$

Lemma-Definition C.4 (*Tensor product of K-endomorphisms*)

If $f_1 : V \longrightarrow V$ and $f_2 : W \longrightarrow W$ are two endomorphisms of finite-dimensional K-vector spaces V and W, then the **tensor product** of f_1 and f_2 is the K-endomorphism $f_1 \otimes f_2$ of $V \otimes_K W$ defined by

$$\begin{aligned} f_1 \otimes f_2 : \quad V \otimes_K W &\longrightarrow V \otimes_K W \\ v \otimes w &\mapsto (f_1 \otimes f_2)(v \otimes w) := f_1(v) \otimes f_2(w)\,. \end{aligned}$$

Furthermore, $\mathrm{Tr}(f_1 \otimes f_2) = \mathrm{Tr}(f_1)\,\mathrm{Tr}(f_2)$.

Proof: It is straightforward to check that $f_1 \otimes f_2$ is K-linear. Then, choosing ordered bases $B_V = (v_1, \dots, v_n)$ and $B_W = (w_1, \dots, w_m)$ of V and W respectively, it is straightforward from the definitions to check that the matrix of $f_1 \otimes f_2$ w.r.t. the basis $B_{V \otimes_K W}$, ordered w.r.t. the lexicographical order, is the Kronecker product of the matrices of f_1 w.r.t. B_V and of f_2 w.r.t. to B_W. The trace formula follows. ■