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Due date: Wednesday, the 7th of June 2023, 17:00 | SS 2023

Throughout, unless otherwise stated, $K = \mathbb{C}$ is the field of complex numbers and $(G, \cdot)$ a finite group with neutral element 1_G. Each Exercise is worth 4 points.

Exercise 1

(a) Exhibit a $\mathbb{C}$-basis of $Cl(G)$ and deduce that $\dim_{\mathbb{C}} Cl(G) = |C(G)|$.

(b) Verify that the form

$$\langle -, - \rangle_G : \mathcal{F}(G, \mathbb{C}) \times \mathcal{F}(G, \mathbb{C}) \longrightarrow \mathbb{C}, (f_1, f_2) \mapsto \langle f_1, f_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

is sesquilinear and Hermitian.

Exercise 2

Let V be a $\mathbb{C}G$-module (i.e. finite-dimensional) with character χ_V. Consider the $\mathbb{C}$-subspace $V^G := \{v \in V \mid g \cdot v = v \; \forall g \in G\}$. Prove that

$$\dim_{\mathbb{C}} V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

in two different ways:

1. considering the scalar product of χ_V with the trivial character $\mathbf{1}_G$;
2. seeing V^G as the image of the projector $\pi : V \longrightarrow V, v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v$.

Exercise 3

(a) Suppose that G is a finite abelian group. Describe all pairwise non-equivalent irreducible $\mathbb{C}$-representations of G.

(b) Fix $n \in \mathbb{Z}_{>0}$ and let $\zeta \in \mathbb{C}$ be a primitive nth root of unity. Use the first orthogonality relations to prove that for all $j \in \mathbb{Z}$,

$$\sum_{i=0}^{n-1} \zeta^{ij} = \begin{cases} n & \text{if } j \equiv 0 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4

(a) Let V be a $\mathbb{C}G$-module (i.e. finite-dimensional), and $W \leq V$ a $\mathbb{C}G$-submodule. We denote by $\chi_V, \chi_W, \chi_{V/W}$ the characters afforded by the $\mathbb{C}G$-modules V, W and V/W respectively. Prove that

$$\chi_V = \chi_W + \chi_{V/W}.$$

(b) Let G_1 and G_2 be two finite groups and $\phi : G_1 \longrightarrow G_2$ a group homomorphism. Let $\chi \in \mathrm{Irr}(G_2)$.

(i) Prove that $\chi \circ \phi$ is a character of G_1, it is called the **restriction of** χ **along** ϕ.

(ii) Prove that if ϕ is an isomorphism, then $\chi \circ \phi$ is irreducible.

(iii) Prove that if ϕ is surjective, then $\chi \circ \phi$ is irreducible.

(iv) Prove that in general $\chi \circ \phi$ is not irreducible.

Exercise 5 (**Bonus Exercise, +2 points**)
Solve again Parts (a)(iii) and (b) of Exercise 4 on Sheet 2 using characters.