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  $$\Longrightarrow \boxed{\chi_i(g_j) = \zeta^{(i-1)j} \quad \forall\, 1 \leq i \leq n,\ \forall\, 0 \leq j \leq n-1}$$

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As 'table':

	1	g	g^2	$\dots$	g^{n-1}
$\chi_1 = 1_G$	1	1	1	$\dots$	1
χ_2	1	φ	φ^2	$\dots$	φ^{n-1}
χ_3	1	φ^2	φ^4	$\dots$	$\varphi^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
χ_n	1	φ^{n-1}	$\varphi^{2(n-1)}$	$\dots$	$\varphi^{(n-1)^2}$

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and $\quad |C_1| = 1, \; |C_2| = 3, \; |C_3| = 2$

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Next, we calculate the characters of G and their values.

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In Example 2(d) we exhibited 3 pairwise non-equivalent irreducible representations of S_3, namely

$$\rho_1 : S_3 \longrightarrow \mathbb{C}^\times$$
$$\sigma \longmapsto 1$$

$$\rho_2 : S_3 \longrightarrow \mathbb{C}^\times$$
$$\sigma \longmapsto \operatorname{sgn}(\sigma)$$

$$\rho_3 : S_3 \longrightarrow GL_2(\mathbb{C})$$
$$(12) \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
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$\Rightarrow$ The character table of S_3 is :

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	Id	(12)	(123)	
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Notice: the degree formula reds

$$\chi_1(1)^2 + \chi_2(1)^2 + \chi_3(1)^2 = 1 + 1 + 4 = 6 = |G|$$

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so, we could also have deduced from this that χ_1, χ_2, χ_3 are <u>all</u> the irreducible characters of S_3.