EXAMPLE

THE CHARACTER TABLE OF S_4

USING INFLATION FROM $S_3 \cong S_4/V_4$

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$$\implies C_1 = \{\underbrace{Id}_{g_1}\},\ C_2 = [\underbrace{(1\,2)}_{g_2}],\ C_3 = [\underbrace{(1\,2\,3)}_{g_3}],\ C_4 = [\underbrace{(12)(34)}_{g_4}],\ C_5 = [\underbrace{(1234)}_{g_5}]$$

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$\implies \quad \boxed{r = |C(G)| = |Irr(G)| = 5}$

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And $|C_1| = 1$, $|C_2| = 6$, $|C_3| = 8$, $|C_4| = 3$, $|C_5| = 6$,
so by the orbit-stabiliser theorem the centraliser orders are

$$\boxed{|C_G(g_1)| = 24, \; |C_G(g_2)| = 4, \; |C_G(g_3)| = 3, \; |C_G(g_4)| = 8, \; |C_G(g_5)| = 4}$$

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Last week, we calculated $X(S_3)$:

	Id	(12)	(123)	
$\chi_1^{S_3}$	1	1	1	(trivial character)
$\chi_2^{S_3}$	1	-1	1	(sign character)
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By Theorem 14.6 we can "inflate" the irreducible characters of $\mathcal{S}_3$ to S_4. We obtain

$$\chi_1 = \mathrm{Inf}_{S_4/V_4}^{S_4}(\chi_1^{S_3}) = 1_{S_4}\,,\ \chi_2 := \mathrm{Inf}_{S_4/V_4}^{S_4}(\chi_2^{S_3}),\ \chi_3 := \mathrm{Inf}_{S_4/V_4}^{S_4}(\chi_3^{S_3}) \in \mathrm{Irr}(S_4)$$

More precisely, we have a part of $X(S_4)$ as follows:

	Id	(12)	(123)	$(12)(34)$	(1234)
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This is because the isomorphism between S_4/V_4 and S_3 maps:

$$
\begin{array}{rcl}
S_4/V_4 & \xleftrightarrow{\cong} & S_3 \\
\mathrm{Id}\,V_4 & \longmapsto & \mathrm{Id} \\
(12)V_4 & \longmapsto & \text{2-cycle} \\
(123)V_4 & \longmapsto & \text{3-cycle} \\
\mathrm{Id}\,V_4 = (12)(34)V_4 & \longmapsto & \mathrm{Id} \\
(12)V_4 = (1234)V_4 & \longmapsto & \text{2-cycle}
\end{array}
$$

(group isomorphisms preserve the orders of elements!)

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$\Rightarrow \chi_4(Id)^2 + \chi_5(Id)^2 = 18$

$\Rightarrow \chi_4(Id) = \chi_5(Id) = 3$ (Only possibility!)

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② 2^{nd} Orthogonality Relations with 3^{rd} column and 3^{rd} column yield:

$$\sum_{i=1}^{5} \chi_i((123)) \underbrace{\overline{\chi_i((123))}}_{\substack{=\chi_i((123)) \\ (\text{since in } \mathbb{R})}} = |C_G((123))| = 3$$

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$$\| $$

$$1+1+1+\chi_4((123))^2+\chi_5((123))^2 \Rightarrow \boxed{\chi_4((123)) = \chi_5((123)) = 0}$$

③ 2nd Orthogonality Relations with | 4th column and 4th column yield:
5th column   5th column

$$\ldots \qquad \left.\begin{array}{l} \chi_4(((12)(34))^2) = \chi_5(((12)(34))^2) = 1 \\ \chi_4((1234)^2) = \chi_5((1234)^2) = 1 \end{array}\right\} \Rightarrow \text{all these entries are } \pm 1$$

④ 2nd Orthogonality Relations with 1st column and 2nd column yield:

$$\chi_4((12)) = 1 \quad \text{and} \quad \chi_5((12)) = -1$$

③ 2^{nd} Orthogonality Relations with $\left|\begin{array}{ll} 4^{th} \text{ column and} & 4^{th} \text{ column} \\ 5^{th} \text{ column} & 5^{th} \text{ column} \end{array}\right.$ yield:

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④ 2^{nd} Orthogonality Relations with 1st column and 2^{nd} column yield:

$$\chi_4((12)) = 1 \quad \text{and} \quad \chi_5((12)) = -1$$

⑤ 1st Orthogonality Relations with 3^{rd} row and 4^{th} row yield:

$$0 = \sum_{k=1}^{5} \frac{1}{|C_G(g_k)|} \chi_3(g_k)\overline{\chi_4(g_k)} = \frac{6}{24} + \frac{1}{4}\chi_4((12)(34))$$

$$\Rightarrow \chi_4((12)(34)) = -1$$

3^{rd} row and 5^{th} row yield: $\chi_5((1234)) = -1$

⑥ 1st Orthogonality Relations with $\left|\begin{array}{l}\text{1st row}\\ \text{1st row}\end{array}\right.$ and $\left|\begin{array}{l}4^{\text{th}}\text{ row}\\ 5^{\text{th}}\text{ row}\end{array}\right.$ yield:

$$\chi_5((12)(34)) = -1 \quad , \quad \chi_4((1234)) = 1$$

⑥ 1st Orthogonality Relations with $\left| \begin{matrix} \text{1st row} \\ \text{1st row} \end{matrix} \right.$ and $\left| \begin{matrix} \text{4th row} \\ \text{5th row} \end{matrix} \right.$ yield:

$$\chi_5((12)(34)) = -1 \quad , \quad \chi_4((1234)) = 1$$

⑦ Conclusion:

$X(S_4) =$

	Id	(12)	(123)	(12)(34)	(1234)
$\chi_1 = 1_{S_4}$	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1