Chapter 6. Induction and Restriction of Characters

In this chapter we present important methods to construct / relate characters of a group, given characters of subgroups or overgroups. The main idea is that we would like to be able to use the character tables of groups we know already in order to compute the character tables of subgroups or overgroups of these groups.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group, $H \leqslant G$ and $N \trianglelefteq G$, $i_H : H \longrightarrow G, h \mapsto h$ is the canonical inclusion of H in G and $\pi_N : G \longrightarrow G/N, g \mapsto gN$ is the quotient morphism;
- $K := \mathbb{C}$ be the field of complex numbers;
- $\mathrm{Irr}(G) := \{\chi_1, \ldots, \chi_r\}$ denote the set of pairwise distinct irreducible characters of G;
- $C_1 = [g_1], \ldots, C_r = [g_r]$ denote the conjugacy classes of G, where $g_1, \ldots, g_r$ is a fixed set of representatives; and
- we use the convention that $\chi_1 = \mathbf{1}_G$ and $g_1 = 1 \in G$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

19 Induction and Restriction

We aim at *inducing* and *restricting* characters from subgroups, resp. overgroups. We start with the operation of induction, which is a subtle operation to construct a class function on G from a given class function on a subgroup $H \leqslant G$. We will focus on characters in a second step.

Definition 19.1 (*Induced class function*)

Let $H \leqslant G$ and $\varphi \in \mathcal{C}l(H)$ be a class function on H. Then the **induction of φ from H to G** is

$$\begin{array}{rccl} \mathrm{Ind}_H^G(\varphi) =: \varphi\uparrow_H^G : & G & \longrightarrow & \mathbb{C} \\ & g & \mapsto & \varphi\uparrow_H^G(g) := \frac{1}{|H|}\sum_{x\in G}\varphi^\circ(xgx^{-1}), \end{array}$$

where for $y \in G$, $\varphi^\circ(y) := \begin{cases} \varphi(y) & \text{if } y \in H, \\ 0 & \text{if } y \notin H. \end{cases}$

Remark 19.2

With the notation of Definition 19.1 the following holds:

(a)
$$\varphi\uparrow_H^G(g) = \frac{1}{|H|}\sum_{x\in G}\varphi^\circ(xgx^{-1}) = \frac{1}{|H|}\sum_{\substack{x\in G\\ xgx^{-1}\in H}}\varphi(xgx^{-1})\,;$$

(b) the function $\varphi\uparrow_H^G$ is a class function on G, because for every $g, y \in G$,
$$\varphi\uparrow_H^G(ygy^{-1}) = \frac{1}{|H|}\sum_{x\in G}\varphi^\circ(xygy^{-1}x^{-1}) \overset{s:=yx}{=} \frac{1}{|H|}\sum_{s\in G}\varphi^\circ(sgs^{-1}) = \varphi\uparrow_H^G(g)\,.$$

In contrast, the operation of restriction is based on the more elementary idea that any map can be restricted to a subset of its domain. For class functions / representations / characters we are essentially interested in restricting these (seen as maps) to subgroups.

Definition 19.3 (*Restricted class function*)

Let $H \leqslant G$ and $\psi \in \mathcal{Cl}(G)$ be a class function on G. Then the **restriction of ψ from G to H** is
$$\mathrm{Res}_H^G(\psi) := \psi\downarrow_H^G := \psi|_H = \psi\circ i_H\,.$$
This is obviously again a class function on H.

Remark 19.4

If ψ is a character of G afforded by the $\mathbb{C}$-representation $\rho : G \longrightarrow \mathrm{GL}(V)$, then clearly $\psi\downarrow_H^G$ is the character afforded by the $\mathbb{C}$-representation $\mathrm{Res}_H^G(\rho) := \rho\downarrow_H^G := \rho|_H = \rho\circ i_H : H \longrightarrow \mathrm{GL}(V)$. See Exercise 9.10(i).

Exercise 19.5

Let $H \leqslant J \leqslant G$ and let $g \leqslant G$. Prove the following assertions:

(a) $\varphi \in \mathcal{Cl}(H) \implies \varphi\uparrow_H^G(g) = \sum_{\substack{Hx\in H\backslash G\\ Hx=Hxg}}\varphi(xgx^{-1})\,;$

(b) $\varphi \in \mathcal{Cl}(H) \implies (\varphi\uparrow_H^J)\uparrow_J^G = \varphi\uparrow_H^G$ (transitivity of induction);

(c) $\psi \in \mathcal{Cl}(G) \implies (\psi\downarrow_J^G)\downarrow_H^J = \psi\downarrow_H^G$ (transitivity of restriction);

(d) the maps
$$\mathrm{Ind}_H^G : \mathcal{Cl}(H) \longrightarrow \mathcal{Cl}(G), \varphi \mapsto \varphi\uparrow_H^G \quad\text{and}\quad \mathrm{Res}_H^G : \mathcal{Cl}(G) \longrightarrow \mathcal{Cl}(H), \psi \mapsto \psi\downarrow_H^G$$
are $\mathbb{C}$-linear;

(e) $\varphi \in \mathcal{Cl}(H)$ and $\psi \in \mathcal{Cl}(G) \implies \psi\cdot\varphi\uparrow_H^G = \left(\psi\downarrow_H^G\cdot\varphi\right)\uparrow_H^G$ (Frobenius formula).

Remark 19.6

We conclude from Exercise 19.5(d) and Corollary 9.8, that the induction and the restriction of a virtual character is again a virtual character. In other words, if $H \leqslant G$, then:

(a) $\varphi \in \mathbb{Z}\operatorname{Irr}(H) \implies \varphi\uparrow_H^G \in \mathbb{Z}\operatorname{Irr}(G)$; and

(b) $\psi \in \mathbb{Z}\operatorname{Irr}(G) \implies \psi\downarrow_H^G \in \mathbb{Z}\operatorname{Irr}(H)$.

Theorem 19.7 (*Frobenius reciprocity*)

Let $H \leqslant G$, let $\varphi \in \mathcal{C}l(H)$ be a class function on H, and let $\psi \in \mathcal{C}l(G)$ be a class function on G. Then,

$$\langle \varphi\uparrow_H^G, \psi\rangle_G = \langle \varphi, \psi\downarrow_H^G\rangle_H \qquad \text{and} \qquad \langle \psi, \varphi\uparrow_H^G\rangle_G = \langle \psi\downarrow_H^G, \varphi\rangle_H .$$

Note: If φ and ψ are characters, then clearly all four numbers are equal.

Proof: Since $\langle -, -\rangle_G$ and $\langle -, -\rangle_H$ are hermitian forms, the 1st equality holds if and only if the 2nd equality holds. Hence, it suffices to prove the second one. By the definitions of the scalar products and of the induction, a direct computation yields:

$$\begin{aligned}
\langle \psi, \varphi\uparrow_H^G\rangle_G = \frac{1}{|G|}\sum_{g\in G}\psi(g)\overline{\varphi\uparrow_H^G(g)} &= \frac{1}{|G|}\sum_{g\in G}\psi(g)\frac{1}{|H|}\sum_{x\in G}\overline{\varphi^\circ(xgx^{-1})} \\
&= \frac{1}{|G|\cdot|H|}\sum_{g\in G}\sum_{\substack{x\in G\\ xgx^{-1}\in H}}\psi(xgx^{-1})\overline{\varphi(xgx^{-1})} \\
&= \frac{1}{|H|}\sum_{s\in H}\psi\downarrow_H^G(s)\overline{\varphi(s)} \\
&= \langle \psi\downarrow_H^G, \varphi\rangle_H ,
\end{aligned}$$

where the third equality comes from the fact that ψ is a class function on G, and for the fourth equality we set $s := xgx^{-1}$. ■

Corollary 19.8

Let $H \leqslant G$ and let χ be a character of H of degree n. Then the induced class function $\chi\uparrow_H^G$ is a character of G of degree $n \cdot |G : H|$.

Proof: Given $\psi \in \operatorname{Irr}(G)$ by Frobenius reciprocity we can set

$$m_\psi := \langle \chi\uparrow_H^G, \psi\rangle_G = \langle \chi, \psi\downarrow_H^G\rangle_H \in \mathbb{Z}_{\geqslant 0} ,$$

which is an integer because both χ and $\psi\downarrow_H^G$ are characters of H. Therefore,

$$\chi\uparrow_H^G = \sum_{\psi\in\operatorname{Irr}(G)} m_\psi \psi$$

is a non-negative integral linear combination of irreducible characters of G, hence a character of G. Moreover,

$$\chi\uparrow_H^G(1) = \frac{1}{|H|}\sum_{x\in G}\chi^\circ(1) = \frac{1}{|H|}|G|\chi(1) = \chi(1)|G : H| .$$

■

Example 11

(a) The restriction of the trivial character of G from G to H is obviously the trivial character of H.

(b) If $H = \{1\}$, then $\mathbf{1}_{\{1\}}\uparrow_{\{1\}}^G = \chi_{\text{reg}}$. Indeed, if $g \in G$ then, it follows from Corollary 10.2 that

$$\mathbf{1}_{\{1\}}\uparrow_{\{1\}}^G (g) = \frac{1}{|\{1\}|}\sum_{x\in G} \underbrace{\mathbf{1}^\circ_{\{1\}}(x^{-1}gx)}_{=0 \text{ unless } g=1} = \delta_{1g}|G| = \chi_{\text{reg}}(g)\,.$$

(c) Let $G = S_3$, $H = \langle(1\ 2)\rangle$, and let $\varphi : H \to \mathbb{C}$ with $\varphi(\text{Id}) = 1$, $\varphi((1\ 2)) = -1$ be the sign homomorphism on H. By the remark, it is enough to compute $\varphi\uparrow_H^G$ on representatives of the conjugacy classes of S_3, e.g. Id, (1 2) and (1 2 3):

$$\varphi\uparrow_H^G (\text{Id}) = \frac{1}{2}\sum_{x\in S_3}\varphi^\circ(\text{Id}) = \frac{1}{2}\cdot|S_3|\cdot 1 = 3\,,$$

$$\varphi\uparrow_H^G ((1\ 2\ 3)) = \frac{1}{2}\sum_{x\in S_3}\varphi^\circ(x^{-1}(1\ 2\ 3)x) = \frac{1}{2}\sum_{x\in S_3} 0 = 0\,,$$

(as the conjugacy class of a 3-cycle contains only 3-cycles and $\varphi(\text{3-cycle}) = 0$)

$$\varphi\uparrow_H^G ((1\ 2)) = \frac{1}{2}\sum_{x\in S_3}\varphi^\circ(x^{-1}(1\ 2)x) = \frac{1}{2}\left(2\varphi^\circ((1\ 2)) + 2\varphi^\circ((1\ 3)) + 2\varphi^\circ((2\ 3))\right) = -1\,.$$

Moreover we see from the character table of S_3 (Example 5) that $\varphi\uparrow_H^G = \chi_2 + \chi_3$. But we can also compute with Frobenius reciprocity, that

$$0 = \langle\varphi, \chi_1\downarrow_H^G\rangle_H = \langle\varphi\uparrow_H^G, \chi_1\rangle_G$$

and similarly

$$1 = \langle\varphi, \chi_2\downarrow_H^G\rangle_H = \langle\varphi\uparrow_H^G, \chi_2\rangle_G \quad \text{and} \quad 1 = \langle\varphi, \chi_3\downarrow_H^G\rangle_H = \langle\varphi\uparrow_H^G, \chi_3\rangle_G\,.$$

Example 12 (*The character table of the alternating group A_5*)

The conjugacy classes of $G = A_5$ are

$$C_1 = \{\text{Id}\}\ ,\ C_2 = [(1\ 2)(3\ 4)]\ ,\ C_3 = [(1\ 2\ 3)]\ ,\ C_4 \cup C_5 = \{\text{5-cycles}\}\,,$$

i.e. $g_1 = \text{Id}, g_2 = (1\ 2)(3\ 4), g_3 = (1\ 2\ 3)$ and $g \in C_4 \Rightarrow o(g) = 5$ and $g^{-1} \in C_4$ but $g^2, g^3 \in C_5$ so that we can choose $g_4 := (1\ 2\ 3\ 4\ 5)$ and $g_5 := (1\ 3\ 5\ 2\ 4)$. This yields:

$$|\operatorname{Irr}(A_5)| = 5 \text{ and } |C_1| = 1,\ |C_2| = 15,\ |C_3| = 20,\ |C_4| = |C_5| = 12\,.$$

We obtain the character table of A_5 as follows:

- We know that the trivial character $\mathbf{1}_G = \chi_1$ is one of the irreducible characters, hence we need to determine $\operatorname{Irr}(A_5)\setminus\{\mathbf{1}_G\} = \{\chi_2, \chi_3, \chi_4, \chi_5\}$.
- Now, $H := A_4 \leqslant A_5$ and we have already computed the character table of A_4 in Exercise Sheet 5. Therefore, inducing the trivial character of A_4 from A_4 to A_5 we obtain that

$$\begin{aligned}
\mathbf{1}_H\!\uparrow_H^G (\mathrm{Id}) &= 1 \cdot |G : H| = 5 \quad \text{(see Cor. 19.8)}\\
\mathbf{1}_H\!\uparrow_H^G \big((1\,2)(3\,4)\big) &= \tfrac{1}{12} \cdot 12 = 1\\
\mathbf{1}_H\!\uparrow_H^G \big((1\,2\,3)\big) &= \tfrac{1}{12} \cdot 24 = 2\\
\mathbf{1}_H\!\uparrow_H^G (5\text{-cycle}) &= \tfrac{1}{12} \cdot 0 = 0
\end{aligned}$$

Now, by Frobenius reciprocity

$$\langle \mathbf{1}_H\!\uparrow_H^G, \chi_1 \rangle_G = \langle \mathbf{1}_H, \underbrace{\chi_1\!\downarrow_H^G}_{=\mathbf{1}_H} \rangle_H = 1\,.$$

It follows (check it) that $\langle \mathbf{1}_H\!\uparrow_H^G - \chi_1, \mathbf{1}_H\!\uparrow_H^G - \chi_1 \rangle_G = 1$, so $\mathbf{1}_H\!\uparrow_H^G - \chi_1$ is an irreducible character, say $\chi_4 := \mathbf{1}_H\!\uparrow_H^G - \chi_1$. The values of χ_4 are given by $(4, 0, 1, -1, -1)$ on C_1, C_2, C_3, C_4, C_5 respectively.

- Next, as A_5 is a non-abelian simple group, we have $A_5/[A_5, A_5] = 1$, and hence the trivial character is the unique linear character of A_5 and $\chi_2(1), \chi_3(1), \chi_5(1) \geqslant 3$. (You have also proved in Exercise 19, Sheet 6 that simple groups do not have irreducible characters of degree 2.) Then the degree formula yields

$$\chi_2(1)^2 + \chi_3(1)^2 + \chi_5(1)^2 = |A_5| - \chi_1(1)^2 - \chi_4(1)^2 = 20 - 1 - 16 = 43\,.$$

As degrees of characters must divide the group order, it follows from this formula that $\chi_2(1), \chi_3(1), \chi_5(1) \in \{3, 4, 5, 6\}$, but then also that it is not possible to have an irreducible character of degree 6. From this we easily see that only possibility, up to relabelling, is $\chi_2(1) = \chi_3(1) = 3$ and $\chi_5(1) = 5$. Hence at this stage, we already have the following part of the character table:

	C_1	C_2	C_3	C_4	C_5
$\|C_k\|$	1	15	20	12	12
$\|C_G(g_k)\|$	60	4	3	5	5
χ_1	1	1	1	1	1
χ_2	3	.	.	.	.
χ_3	3	.	.	.	.
χ_4	4	0	1	-1	-1
χ_5	5	.	.	.	.

- Next, we have that

$$\gcd(\chi_2(1), |C_3|) = \gcd(\chi_3(1), |C_3|) = \gcd(\chi_5(1), |C_4|) = \gcd(\chi_5(1), |C_5|) = 1\,,$$

so that the corresponding character values must all be zero by Corollary 17.7 and we get:

	C_1	C_2	C_3	C_4	C_5
$\|C_k\|$	1	15	20	12	12
$\|C_G(g_k)\|$	60	4	3	5	5
χ_1	1	1	1	1	1
χ_2	3	.	0	.	.
χ_3	3	.	0	.	.
χ_4	4	0	1	-1	-1
χ_5	5	.	.	0	0

- Applying the Orthogonality Relations yields:
  1st, 3rd column $\Rightarrow \chi_5(g_3) = -1$ and the scalar product $\langle \chi_1, \chi_5 \rangle_G = 0 \Rightarrow \chi_5(g_2) = 1$.

- Finally, to fill out the remaining gaps, we can induce from the cyclic subgroup $Z_5 := \langle (1\ 2\ 3\ 4\ 5) \rangle \leqslant A_5$. Indeed, choosing the non-trivial irreducible character ψ of Z_5 which was denoted "χ_3" in Example 4 gives

  $$\psi \uparrow_{Z_5}^{G} = (12, 0, 0, \zeta^2 + \zeta^3, \zeta + \zeta^4)$$

  where $\zeta = \exp(2\pi\mathbf{i}/5)$ is a primitive 5-th root of unity. Then we compute that

  $$\langle \psi \uparrow_{Z_5}^{G}, \chi_4 \rangle_G = 1 = \langle \psi \uparrow_{Z_5}^{G}, \chi_5 \rangle_G \quad \Longrightarrow \quad \psi \uparrow_{Z_5}^{G} - \chi_4 - \chi_5 = (3, -1, 0, -\zeta - \zeta^4, -\zeta^2 - \zeta^3)$$

  and this character must be irreducible, because it is not the sum of 3 copies of the trivial character. Hence we set $\chi_2 := \psi \uparrow_{Z_5}^{G} - \chi_4 - \chi_5$ and the values of χ_3 then easily follow from the 2nd Othogonality Relations:

	C_1	C_2	C_3	C_4	C_5
$\lvert C_k \rvert$	1	15	20	12	12
$\lvert C_G(g_k) \rvert$	60	4	3	5	5
χ_1	1	1	1	1	1
χ_2	3	-1	0	$-\zeta - \zeta^4$	$-\zeta^2 - \zeta^3$
χ_3	3	-1	0	$-\zeta^2 - \zeta^3$	$-\zeta - \zeta^4$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

Remark 19.9 (*Induction of $\mathbb{C}H$-modules*)

If you have attended the lecture *Commutative Algebra* you have studied the *tensor product of modules*. In the M.Sc. lecture *Representation Theory* you will see that induction of modules is defined through a tensor product, extending the scalars from $\mathbb{C}H$ to $\mathbb{C}G$. More precisely, if M is a $\mathbb{C}H$-module, then the induction of M from H to G is defined to be $\mathbb{C}G \otimes_{\mathbb{C}H} M$. Moreover, if M affords the character χ, then $\mathbb{C}G \otimes_{\mathbb{C}H} M$ affords the character $\chi \uparrow_H^G$.

20 Clifford Theory

Clifford theory is a generic term for a series of results relating the representation / character theory of a given group G to that of a normal subgroup $N \trianglelefteq G$ through induction and restriction.

Notation 20.1

If $H \leqslant G$ and $x \in G$, then we let

$$\begin{array}{rccc} c_x : & H & \longrightarrow & xHx^{-1} \\ & h & \mapsto & xhx^{-1} \end{array}$$

denote the conjugation homomorphism by x.