Theorem 5.6 (Schur's Relations)

Assume $\mathrm{char}(K) \nmid |G|$. Let $Q : G \longrightarrow \mathrm{GL}_n(K)$ and $P : G \longrightarrow \mathrm{GL}_m(K)$ be irreducible matrix representations.

(a) If $P \not\sim Q$, then $\frac{1}{|G|}\sum_{g\in G} P(g)_{ri}Q(g^{-1})_{js} = 0$ for all $1 \leqslant r, i \leqslant m$ and all $1 \leqslant j, s \leqslant n$.

(b) If $K = \overline{K}$ and $\mathrm{char}(K) \nmid n$, then $\frac{1}{|G|}\sum_{g\in G} Q(g)_{ri}Q(g^{-1})_{js} = \frac{1}{n}\delta_{ij}\delta_{rs}$ for all $1 \leqslant r, i, j, s \leqslant n$.

Proof: Set $V := K^n$, $W := K^m$ and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : G \longrightarrow \mathrm{GL}(W)$ be the K-representations induced by Q and P, respectively, as defined in Remark 1.2. Furthermore, consider the K-linear map $\psi : V \longrightarrow W$ whose matrix with respect to the standard bases of $V = K^n$ and $W = K^m$ is the elementary matrix

$$i\begin{bmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{bmatrix}_{\substack{\\ j}} =: E_{ij} \in M_{m\times n}(K)$$

(i.e. the unique nonzero entry of E_{ij} is its (i,j)-entry).

(a) By Proposition 5.5(a),

$$\widetilde{\psi} = \frac{1}{|G|}\sum_{g\in G} \rho_W(g)\circ\psi\circ\rho_V(g^{-1}) = 0$$

because $P \not\sim Q$, and hence $\rho_V \not\sim \rho_W$. In particular the (r,s)-entry of the matrix of $\widetilde{\psi}$ with respect to the standard bases of $V = K^n$ and $W = K^m$ is zero. Thus,

$$0 = \frac{1}{|G|}\sum_{g\in G}\left[P(g)E_{ij}Q(g^{-1})\right]_{rs} = \frac{1}{|G|}\sum_{g\in G} P(g)_{ri}\cdot 1\cdot Q(g^{-1})_{js}$$

because the unique nonzero entry of the matrix E_{ij} is its (i,j)-entry.

(b) Now we assume that $P = Q$, and hence $n = m$, $V = W$, $\rho_V = \rho_W$. Then by Proposition 5.5(b),

$$\widetilde{\psi} := \frac{1}{|G|}\sum_{g\in G} \rho_V(g)\circ\psi\circ\rho_V(g^{-1}) = \frac{\mathrm{Tr}(\psi)}{n}\cdot \mathrm{Id}_V = \begin{cases} \frac{1}{n}\cdot \mathrm{Id}_V & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Therefore the (r,s)-entry of the matrix of $\widetilde{\psi}$ with respect to the standard basis of $V = K^n$ is

$$\frac{1}{|G|}\sum_{g\in G}\left[Q(g)E_{ij}Q(g^{-1})\right]_{rs} = \begin{cases} \left(\frac{1}{n}\cdot \mathrm{Id}_V\right)_{rs} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Again, because the unique nonzero entry of the matrix E_{ij} is its (i,j)-entry, it follows that

$$\frac{1}{|G|}\sum_{g\in G} Q(g)_{ri}Q(g^{-1})_{js} = \frac{1}{n}\delta_{ij}\delta_{rs}\,.$$

■

6 Representations of Finite Abelian Groups

In this section we give an immediate application of Schur's Lemma encoding the representation theory of finite abelian groups over an algebraically closed field K whose characteristic is coprime to the order of the group.

Proposition 6.1

Assume G is a finite <u>abelian</u> group, $K = \overline{K}$ and $\mathrm{char}(K) \nmid |G|$. Then the K-dimension of any simple KG-module is equal to 1.
(Equivalently, any irreducible K-representation of G has degree 1.)

Proof: Let V be a simple KG-module, and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be the underlying K-representation (i.e. as given by Proposition 4.3).

<u>Claim:</u> any K-subspace of V is in fact a KG-submodule.

<u>Proof:</u> Fix $g \in G$ and consider $\rho_V(g)$. By definition $\rho_V(g) \in \mathrm{GL}(V)$, hence it is a K-linear endomorphism of V. We claim that it is in fact KG-linear. Indeed, as G is abelian, $\forall\, h \in G$, $\forall\, v \in V$ we have

$$\begin{aligned}\rho_V(g)(h \cdot v) = \rho_V(g)\big(\rho_V(h)(v)\big) &= \big[\rho_V(g)\rho_V(h)\big](v)\\ &= \big[\rho_V(gh)\big](v)\\ &= \big[\rho_V(hg)\big](v)\\ &= \big[\rho_V(h)\rho_V(g)\big](v)\\ &= \rho_V(h)\big(\rho_V(g)(v)\big)\\ &= h \cdot \big(\rho_V(g)(v)\big)\end{aligned}$$

and it follows that $\rho_V(g)$ is KG-linear, i.e. $\rho_V(g) \in \mathrm{End}_{KG}(V)$. Now, because K is algebraically closed, by part (b) of Schur's Lemma, there exists $\lambda_g \in K$ (depending on g) such that

$$\rho_V(g) = \lambda_g \cdot \mathrm{Id}_V \,.$$

As this holds for every $g \in G$, it follows that any K-subspace of V is G-invariant, which in terms of KG-modules means that any K-subspace of V is a KG-submodule of V.

To conclude, as V is simple, we deduce from the Claim that the K-dimension of V must be equal to 1. ■

Theorem 6.2 (Diagonalisation Theorem)

Assume $K = \overline{K}$ and $\mathrm{char}(K) \nmid |G|$. Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a K-representation of an arbitrary finite group G. Fix $g \in G$. Then, there exists an ordered K-basis B of V with respect to which

$$\big(\rho(g)\big)_B = \begin{bmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon_n \end{bmatrix},$$

where $n := \dim_K(V)$ and each ε_i ($1 \leqslant i \leqslant n$) is an $o(g)$-th root of unity in K.

Proof: Consider the restriction of ρ to the cyclic subgroup generated by g, that is the representation

$$\rho|_{\langle g \rangle} : \langle g \rangle \longrightarrow \mathrm{GL}(V)\,.$$

By Corollary 3.6 to Maschke's Theorem, we can decompose the representation $\rho|_{\langle g \rangle}$ into a direct sum of irreducible K-representations, say

$$\rho|_{\langle g \rangle} = \rho_{V_1} \oplus \cdots \oplus \rho_{V_n}\,,$$

where $V_1, \ldots, V_n \subseteq V$ are $\langle g \rangle$-invariant. Since $\langle g \rangle$ is abelian $\dim_K(V_i) = 1$ for each $1 \leqslant i \leqslant n$ by Proposition 6.1. Now, if for each $1 \leqslant i \leqslant n$ we choose a K-basis $\{x_i\}$ of V_i, then there exist $\varepsilon_i \in K$

$(1 \leqslant i \leqslant n)$ such that $\rho_{V_i}(g) = \varepsilon_i$ and $B := (x_1, \ldots, x_n)$ is an ordered K-basis of V such that

$$\big(\rho(g)\big)_B = \begin{bmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon_n \end{bmatrix}.$$

Finally, as $g^{o(g)} = 1_G$, it follows that for each $1 \leqslant i \leqslant n$,

$$\varepsilon_i^{o(g)} = \rho_{V_i}(g)^{o(g)} = \rho_{V_i}(g^{o(g)}) = \rho_{V_i}(1_G) = 1_K$$

and hence ε_i is an $o(g)$-th root of unity. ■

Scholium 6.3

Assume $K = \overline{K}$, $\mathrm{char}(K) \nmid |G|$ and G is abelian. If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a K-representation of G, then the K-endomorphisms $\rho(g) : V \longrightarrow V$ with g running through G are simultaneously diagonalisable.

Proof: Same argument as in the previous proof, where we may replace "$\langle g \rangle$" with the whole of G. ■

Chapter 3. Characters of Finite Groups

We now introduce the concept of a ***character*** of a finite group. These are functions $\chi : G \longrightarrow \mathbb{C}$, obtained from the representations of the group G by taking traces. Characters have many remarkable properties, and they are the fundamental tools for performing computations in representation theory. They encode a lot of information about the group itself and about its representations in a more compact and efficient manner.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group;
- $K := \mathbb{C}$ be the field of complex numbers; and
- V denote a $\mathbb{C}$-vector space such that $\dim_{\mathbb{C}}(V) < \infty$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

7 Characters

Definition 7.1 (*Character, linear character*)

Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation. The **character** of ρ_V is the $\mathbb{C}$-valued function

$$\begin{array}{rccl} \chi_V : & G & \longrightarrow & \mathbb{C} \\ & g & \mapsto & \chi_V(g) := \mathrm{Tr}\left(\rho_V(g)\right) \end{array}.$$

We also say that ρ_V (or the associated $\mathbb{C}G$-module V) **affords** the character χ_V. The **degree** of χ_V is the degree of ρ_V. If the degree of χ_V is one, then χ_V is called a **linear** character.

Remark 7.2

(a) Recall that in *linear algebra* (see GDM) the trace of a linear endomorphism φ may be concretely computed by taking the trace of the matrix of φ in a chosen basis of the vector space, and this is independent of the choice of the basis.

Thus to compute characters: choose an ordered basis B of V and obtain $\forall\, g \in G$:

$$\chi_V(g) = \mathrm{Tr}\left(\rho_V(g)\right) = \mathrm{Tr}\left(\left(\rho_V(g)\right)_B\right)$$

(b) For a matrix representation $R : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$, the character of R is then

$$\chi_R : \begin{array}{ccc} G & \longrightarrow & \mathbb{C} \\ g & \mapsto & \chi_R(g) := \mathrm{Tr}\left(R(g)\right) \end{array} .$$

Example 3

The character of the trivial representation of G is the function $\mathbf{1}_G : G \longrightarrow \mathbb{C}, g \mapsto 1$ and is called **the trivial character** of G.

Lemma 7.3

Equivalent $\mathbb{C}$-representations afford the same character.

Proof: If $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : G \longrightarrow \mathrm{GL}(W)$ are two $\mathbb{C}$-representations, and $\alpha : V \longrightarrow W$ is an isomorphism of representations, then

$$\rho_W(g) = \alpha \circ \rho_V(g) \circ \alpha^{-1} \quad \forall\, g \in G .$$

Now, by the properties of the trace (GDM) for any two $\mathbb{C}$-endomorphisms β, γ of V we have $\mathrm{Tr}(\beta \circ \gamma) = \mathrm{Tr}(\gamma \circ \beta)$, hence for every $g \in G$ we have

$$\chi_W(g) = \mathrm{Tr}\left(\rho_W(g)\right) = \mathrm{Tr}\left(\alpha \circ \rho_V(g) \circ \alpha^{-1}\right) = \mathrm{Tr}\left(\rho_V(g) \circ \underbrace{\alpha^{-1} \circ \alpha}_{=\mathrm{Id}_V}\right) = \mathrm{Tr}\left(\rho_V(g)\right) = \chi_V(g) .$$

■

Terminology / Notation 7.4

- Again, we allow ourselves to transport terminology from representations to characters. For example, if ρ_V is irreducible (faithful, ...), then the character χ_V is also called **irreducible** (**faithful**, ...).
- We define $\mathrm{Irr}(G)$ to be the set of all irreducible characters of G, and $\mathrm{Lin}(G)$ to be the set of all linear characters of G. (We will see below that $\mathrm{Irr}(G)$ is a finite set.)

Properties 7.5 (*Elementary properties*)

Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation and let $g \in G$. Then the following assertions hold:

(a) $\chi_V(1_G) = \dim_{\mathbb{C}} V$;

(b) $\chi_V(g) = \varepsilon_1 + \ldots + \varepsilon_n$, where $\varepsilon_1, \ldots, \varepsilon_n$ are $o(g)$-th roots of unity in $\mathbb{C}$ and $n = \dim_{\mathbb{C}} V$;

(c) $|\chi_V(g)| \leqslant \chi_V(1_G)$;

(d) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$;

(e) if $\rho_V = \rho_{V_1} \oplus \rho_{V_2}$ is the direct sum of two subrepresentations, then $\chi_V = \chi_{V_1} + \chi_{V_2}$.

Proof:

(a) We have $\rho_V(1_G) = \mathrm{Id}_V$ since representations are group homomorphisms, hence $\chi_V(1_G) = \dim_{\mathbb{C}} V$.

(b) This follows directly from the diagonalisation theorem (Theorem 6.2).

(c) By (b) we have $\chi_V(g) = \varepsilon_1 + \ldots + \varepsilon_n$, where $\varepsilon_1, \ldots, \varepsilon_n$ are roots of unity in $\mathbb{C}$. Hence, applying the triangle inequality repeatedly, we obtain that

$$|\chi_V(g)| = |\varepsilon_1 + \ldots + \varepsilon_n| \leqslant \underbrace{|\varepsilon_1|}_{=1} + \ldots + \underbrace{|\varepsilon_n|}_{=1} = \dim_{\mathbb{C}} V \overset{(a)}{=} \chi_V(1_G)\,.$$

(d) Again by the diagonalisation theorem, there exists an ordered $\mathbb{C}$-basis B of V and $o(g)$-th roots of unity $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{C}$ such that

$$\big(\rho_V(g)\big)_B = \begin{bmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon_n \end{bmatrix}.$$

Therefore

$$\big(\rho_V(g^{-1})\big)_B = \begin{bmatrix} \varepsilon_1^{-1} & 0 & \cdots & 0 \\ 0 & \varepsilon_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon_n^{-1} \end{bmatrix} = \begin{bmatrix} \overline{\varepsilon_1} & 0 & \cdots & 0 \\ 0 & \overline{\varepsilon_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \overline{\varepsilon_n} \end{bmatrix}$$

and it follows that $\chi_V(g^{-1}) = \overline{\varepsilon_1} + \ldots + \overline{\varepsilon_n} = \overline{\varepsilon_1 + \ldots + \varepsilon_n} = \overline{\chi_V(g)}\,.$

(e) For $i \in \{1, 2\}$ let B_i be an ordered $\mathbb{C}$-basis of V_i and consider the $\mathbb{C}$-basis $B := B_1 \sqcup B_2$ of V. Then, by Remark 3.2 for every $g \in G$ we have

$$\big(\rho_V(g)\big)_B = \left[\begin{array}{c|c} \big(\rho_{V_1}(g)\big)_{B_1} & 0 \\ \hline 0 & \big(\rho_{V_2}(g)\big)_{B_2} \end{array}\right],$$

hence $\chi_V(g) = \mathrm{Tr}\big(\rho_V(g)\big) = \mathrm{Tr}\big(\rho_{V_1}(g)\big) + \mathrm{Tr}\big(\rho_{V_2}(g)\big) = \chi_{V_1}(g) + \chi_{V_2}(g)\,.$ ■

Corollary 7.6

Any character of G is a sum of irreducible characters of G.

Proof: By Corollary 3.6 to Maschke's theorem, any $\mathbb{C}$-representation can be written as the direct sum of irreducible subrepresentations. Thus the claim follows from Properties 7.5(e). ■

Notation 7.7

Recall from group theory (*Einführung in die Algebra*) that a group G *acts on itself by conjugation* via

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (g, x) & \mapsto & gxg^{-1} =: {}^g x\,. \end{array}$$

The orbits of this action are the *conjugacy classes* of G, we denote them by $[x] := \{\,{}^g x \mid g \in G\}$, and we write $C(G) := \{[x] \mid x \in G\}$ for the set of all conjugacy classes of G.

The stabiliser of $x \in G$ is its *centraliser* $C_G(x) = \{g \in G \mid {}^g x = x\}$ and the orbit-stabiliser theorem yields

$$|C_G(x)| = \frac{|G|}{|[x]|}\,.$$

Moreover, a function $f : G \longrightarrow \mathbb{C}$ which is constant on each conjugacy class of G, i.e. such that $f(gxg^{-1}) = f(x)\ \forall\, g, x \in G$, is called a **class function** (on G).

Lemma 7.8

Characters are class functions.

Proof: Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation and let χ_V be its character. Again, because by the properties of the trace we have $\mathrm{Tr}(\beta\circ\gamma) = \mathrm{Tr}(\gamma\circ\beta)$ for all $\mathbb{C}$-endomorphisms β, γ of V (GDM !), it follows that for all $g, x \in G$,

$$\begin{aligned}\chi_V(gxg^{-1}) = \mathrm{Tr}\left(\rho_V(gxg^{-1})\right) &= \mathrm{Tr}\left(\rho_V(g)\rho_V(x)\rho_V(g)^{-1}\right)\\ &= \mathrm{Tr}\big(\rho_V(x)\underbrace{\rho_V(g)^{-1}\rho_V(g)}_{=\mathrm{Id}_V}\big) = \mathrm{Tr}\left(\rho_V(x)\right) = \chi_V(x)\,.\end{aligned}$$

■

Exercise 7.9

Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation and let χ_V be its character. Prove the following statements.

(a) If $g \in G$ is conjugate to g^{-1}, then $\chi_V(g) \in \mathbb{R}$.

(b) If $g \in G$ is an element of order 2, then $\chi_V(g) \in \mathbb{Z}$ and $\chi_V(g) \equiv \chi_V(1) \pmod 2$.

Exercise 7.10 (The dual representation / the dual character)

Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation.

(a) Prove that:
  (i) the dual space $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is endowed with the structure of a $\mathbb{C}G$-module via
  $$\begin{array}{ccc} G\times V^* & \longrightarrow & V^* \\ (g,f) & \mapsto & g.f \end{array}$$
  where $(g.f)(v) := f(g^{-1}v)\ \forall\, v \in V$;
  (ii) the character of the associated $\mathbb{C}$-representation ρ_{V^*} is then $\chi_{V^*} = \overline{\chi_V}$; and
  (iii) if ρ_V decomposes as a direct sum $\rho_{V_1} \oplus \rho_{V_2}$ of two subrepresentations, then ρ_{V^*} is equivalent to $\rho_{V_1^*} \oplus \rho_{V_2^*}$.

(b) Determine the duals of the 3 irreducible representations of S_3 given in Example 2(d).

8 Orthogonality of Characters

We are now going to make use of results from the linear algebra (GDM) on the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on G in order to develop further fundamental properties of characters.