Jun.-Prof. Dr. Caroline Lassueur | **FB Mathematik**
Dr. Niamh Farrell

Due date: **Tuesday, 19th of November 2019, 6 p.m.** | WS 2019/20

Exercise 8.
Prove that $J(\mathbb{Z}) = 0$, but not all $\mathbb{Z}$-modules are semisimple.

Exercise 9.
Let K be a commutative ring and A be a K-algebra. Prove that:

(a) $Z(A)$ is a K-subalgebra of A.

(b) If K is a field and $A \neq 0$, then $K \longrightarrow Z(A), \lambda \mapsto \lambda 1_A$ is an injective A-homomorphism.

(c) Prove that if $A = M_n(K)$, then $Z(A) \cong KI_n$, i.e. the K-subalgebra of scalar matrices. (Hint: use the elementary matrices.)

(c) Assume A is the algebra of 2×2 upper-triangular matrices over K. Prove that $Z(A) = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix} \right) \mid a \in K \right\}$.

Exercise 10.
Let K be a field and let $A \neq 0$ be a finite-dimensional K-algebra. The aim of this exercise is to prove that $J(A)$ is the unique maximal nilpotent left ideal of A and $J(Z(A)) = J(A) \cap Z(A)$. Proceed as follows:

(a) Prove that there exists $n \in \mathbb{Z}_{>0}$ such that $J(A)^n = J(A)^{n+1}$. (Hint: consider dimensions.)

(b) Apply Nakayama's Lemma to deduce that $J(A)^n = 0$ and conclude that $J(A)$ is nilpotent.

(c) Prove that if I is an arbitrary nilpotent left ideal of A, then $I \subseteq J(A)$. (Hint: here you should see $J(A)$ as the intersection of the annihilators of the simple A-modules.)

(d) Use the nilpotency of the jacobson radical (of both A and $Z(A)$) to prove that

$$J(Z(A)) = J(A) \cap Z(A)\,.$$

Exercise 11.
Let R be a semisimple ring. Prove the following statements.

(a) Every non-zero left ideal I is generated by an **idempotent** of R, in other words $\exists\, e \in R$ such that $e^2 = e$ and $I = Re$. (Hint: choose a complement I' for I, so that $R^\circ = I \oplus I'$ and write $1 = e + e'$ with $e \in I$ and $e' \in I'$. Prove that $I = Re$.)

(b) If I is a non-zero left ideal of R, then every morphism in $\mathrm{Hom}_R(I, R^\circ)$ is given by right multiplication with an element of R.

(c) If $e \in R$ is an idempotent, then $\mathrm{End}_R(Re) \cong (eRe)^{\mathrm{op}}$ (the opposite ring) as rings via the map $f \mapsto ef(e)e$. In particular $\mathrm{End}_R(R^\circ) \cong R^{\mathrm{op}}$ via $f \mapsto f(1)$.

(d) A left ideal Re generated by an idempotent e of R is minimal (i.e. simple as an R-module) if and only if eRe is a division ring. (Hint: Use Schur's Lemma.)

(e) Every simple left R-module is isomorphic to a minimal left ideal in R.