Throughout G denotes a finite group.

Exercise 12.
Let G be a finite group and let K be a commutative ring. Prove that the regular representation ρ_{reg} of G corresponds to the regular KG-module KG° (i.e. via Proposition 15.3).

Exercise 13.
Let $G := C_2 \times C_2 \times C_2$ and let K be a field such that $\mathrm{char}(K) \neq 2$. Find all the simple KG-modules, up to isomorphism.

Exercise 14.
The aim of this exercise is to prove that if K is a field of positive characteristic p and G is p-group, then $I(KG) = J(KG)$. Proceed as follows:

(a) (Facultative. You can accept this result and treat (b), (c) and (d) only.)
Recall that an ideal I of a ring R is called a **nil ideal** if each element of I is nilpotent. Prove that if I is a nil left ideal in a left Artinian ring R then I is nilpotent.

(b) Prove that $g - 1$ is a nilpotent element for each $g \in G \setminus \{1\}$ and deduce that $I(KG)$ is a nil ideal of KG.

(c) Deduce from (a) and (b) that $I(KG) \subseteq J(KG)$ using Exercise 10 on Exercise Sheet 2.

(d) Conclude that $I(KG) = J(KG)$ using Proposition-Definition 15.7.

Exercise 15.
Let p be a prime number, let $G := C_p = \langle g \mid g^p = 1\rangle$ and let $K := \mathbb{F}_p$. Let $B := (e_1, e_2)$ be the standard ordered basis of $V := K^2$. Consider the KG-module structure on V given by the matrix representation

$$\begin{array}{rccc} X\colon & G & \longrightarrow & GL_2(K) \\ & g^b & \mapsto & \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \end{array}$$

Prove that $W := Ke_1$ is a simple KG-submodule of V. However, there exists no direct sum decomposition of V into simple KG-submodules.

Exercise 16 (**Proof of the Converse of Maschke's Theorem for** $K = \overline{K}$).
Assume $K = \overline{K}$ is an algebraically closed field of characteristic p such that $p \mid |G|$. Set $T := \langle \sum_{g\in G} g\rangle_K$.

(a) Prove that we have a series of KG-submodules given by $KG^\circ \supsetneq I(KG) \supseteq T \supsetneq 0$.

(b) Deduce that KG° has at least two composition factors isomorphic to the trivial module K.

(c) Deduce that KG is not a semisimple K-algebra using Theorem 13.2.