Throughout G denotes a finite group and K a field. All KG-modules considered are assumed to be finitely generated.

Exercise 25.
Let $H, L \leq G$, let M be a KL-module and let N be a KH-module. Use the Mackey formula to prove that:

(a) $M\uparrow_L^G \otimes_K N\uparrow_H^G \cong \bigoplus_{g\in[H\backslash G/L]} ({}^gM\downarrow^{{}^gL}_{H\cap {}^gL} \otimes_K N\downarrow^H_{H\cap {}^gL})\uparrow^G_{H\cap {}^gL}$;

(b) $\mathrm{Hom}_K(M\uparrow_L^G, N\uparrow_H^G) \cong \bigoplus_{g\in[H\backslash G/L]} (\mathrm{Hom}_K({}^gM\downarrow^{{}^gL}_{H\cap {}^gL}, N\downarrow^H_{H\cap {}^gL}))\uparrow^G_{H\cap {}^gL}$.

Exercise 26.
Let M be a KG-module. Prove that the following KG-submodules of M are equal:

(1) $\mathrm{soc}(M)$;

(2) the largest semisimple KG-submodule of M;

(3) $\{m \in M \mid J(KG)m = 0\}$.

Exercise 27.
Prove that:

(a) If P is a projective KG-module, then so is P^*.

(b) If $H \leq G$ and P is a projective KH-module, then $P\uparrow_H^G$ is a projective KG-module.

(c) If P is a projective KG-module and M is an arbitrary KG-module, then $P \otimes_K M$ is projective.

(d) If P is a projective indecomposable KG-module, then $\mathrm{soc}(P)$ is simple. (Hint: consider duals.)

Exercise 28.
Let S be a simple KG-module and let P_S denote the corresponding PIM (i.e. $P_S/\operatorname{rad}(P_S) \cong S$). Let M be an arbitrary KG-module. Prove that:

(a) If T is a simple KG-module then

$$\dim_K \mathrm{Hom}_{KG}(P_S, T) = \begin{cases} \dim_K \mathrm{End}_{KG}(S) & \text{if } S \cong T, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The multiplicity of S as a composition factor of M is

$$\dim_K \mathrm{Hom}_{KG}(P_S, M)/\dim_K \mathrm{End}_{KG}(S).$$