Proseminar
Endliche Coxeter-Gruppen
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Contents

Chapter 1. Definitions and Examples

The aim of this chapter is to introduce Coxeter groups in all generality, consider some important examples, and give a first description of the finite ones. In the next chapters we will give a formal proof of their classification.

References:

[Hum90] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics Press, vol. 29, Cambridge University Press, Cambridge, 1990.

1 Coxeter systems

Coxeter groups are groups defined by a presentation as follows.

Definition 1.1 (*Coxeter system*)

*A **Coxeter system** is a pair (W, S) such that*

(a) *W is a group;*

(b) *$S = \{s_1, \dots, s_n\}$ ($n \in \mathbb{Z}_{>0}$) is a* finite *set of generators for W; and*

(c) *W admits the presentation*

$$W = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \; \forall \, i \leqslant j \rangle ,$$

where $m_{ii} = 1$ for each $1 \leqslant i \leqslant n$, and $m_{ij} \in \{2, 3, \dots, \infty\}$ if $i < j$.

Remark 1.2

Below are some elementary consequences of Definition 1.1.

(1) *$m_{ii} = 1 \Rightarrow s_i^2 = 1$ for each $1 \leqslant i \leqslant n$. As we may assume that $s_1, \dots, s_n \neq 1$, all the generators $s_i \in S$ have order 2 and $s_i^{-1} = s_i$.*

(2) *We refer to W itself as a **Coxeter group** if the underlying above presentation is implicitly understood.*

(3) *If $i < j$, then $(s_i s_j)^{m_{ij}} = 1$ by definition and conjugation by s_j yields*

$$1 = s_j 1 s_j = s_j (s_i s_j)^{m_{ij}} s_j = (s_j s_i)^{m_{ij}} \underbrace{s_j s_j}_{=1} = (s_j s_i)^{m_{ij}} .$$

Thus we may set $m_{ji} := m_{ij}$ and the relation $(s_j s_i)^{m_{ji}} = 1$ holds as well, but is superfluous.

(4) $m_{ij} = 2 \iff 1 = (s_i s_j)^2 = s_i s_j s_i s_j = s_i s_j s_i^{-1} s_j^{-1} = [s_i, s_j] \iff s_i$ *and* s_j *commute.*

(5) If m_{ij} is even, then $(s_i s_j)^{m_{ij}/2} = (s_j s_i)^{m_{ij}/2}$; and
if m_{ji} is odd, then $\underbrace{s_i s_j s_i s_j \cdots s_j s_i}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j s_i \cdots s_i s_j}_{m_{ij} \text{ terms}}$.

(6) We will prove that m_{ij} is precisely the order of $s_i s_j$.

(7) By the above $M := (m_{ij})_{1 \leqslant i,j \leqslant n}$ is a symmetric matrix with all diagonal entries equal to 1. *This matrix is called the* ***Coxeter matrix*** *associated to the Coxeter system (W, S).*

2 Coxeter graphs

Henceforth, by *graph*, we understand a pair (S, A), where S is a finite set and A is a subset of $\mathcal{P}(S)$ consisting of 2-element subsets of S. The elements of S are the *vertices* of the graph and the elements of A are the *edges* of the graph. Furthermore, a *weighted graph* is a pair (G, φ), where $G = (S, A)$ is a graph and $\varphi : A \longrightarrow \mathbb{Z}_{>0} \cup \{\infty\}$ is a map. The valies of φ are the *weights* associated of the edges.

Definition 2.1 (*Coxeter graph*)

The ***Coxeter graph*** *associated to a Coxeter system (W, S) with $S = \{s_1, \ldots, s_n\}$ and Coxeter matrix $(m_{ij})_{1 \leqslant i,j \leqslant n}$ is the weighted graph having S as set of vertices and edges defined and weighted as follows:*

(i) *if $m_{ij} \in \{1, 2\}$ there is no edge between s_i and s_j, and*

(ii) *if $m_{ij} \geqslant 3$ there is an edge between s_i and s_j with weight m_{ij}.*

Moreover, by convention, the weight of an edge is written above it, unless the weight is 3, *in which case it is always omitted.*

Example 1 (*The Coxeter graph F_4*)

The Coxeter group
$W = \langle s_1, s_2, s_3, s_4 \mid s_1^2 = s_2^2 = s_3^2 = s_4^2 = 1, (s_1 s_2)^3 = 1, (s_1 s_3)^2 = 1, (s_1 s_4)^2 = 1, (s_2 s_4)^2 = 1, (s_2 s_3)^4 = 1, (s_3 s_4)^3 = 1 \rangle$
yields the following Coxeter graph and Coxeter matrix:

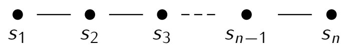
$$\bullet \!-\!\!-\! \bullet \overset{4}{\!-\!\!-\!} \bullet \!-\!\!-\! \bullet \qquad \textit{and} \qquad \begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{pmatrix}$$

Remark 2.2

The data contained in the Coxeter system (W, S) is equivalent to the data contained in the associated Coxeter matrix and equivalent to the data contained in the associated Coxeter graph. If a Coxeter graph G is given, then we denote by $W(G)$ the associated Coxeter group.

Example 2 (*The Coxeter graph A_n ($n \geqslant 2$)*)

The Coxeter graph

A_n 

yields the Coxeter group

$$W(A_n) = \langle s_1, \ldots, s_n \mid s_i^2 = 1 \,\forall\, 1 \leqslant i \leqslant n, (s_i s_j)^2 = 1 \text{ if } i \leqslant j-2, (s_i s_{i+1})^3 = 1 \,\forall\, 1 \leqslant i \leqslant n-1 \rangle$$

and the map

$$\begin{array}{ccc} W(A_n) & \longrightarrow & \mathfrak{S}_{n+1} \\ s_i & \mapsto & (i \;\; i+1) \end{array}$$

defines a group isomorphism between $W(A_n)$ and the symmetric group of degree $n+1$. (Give a proof, if time permits. In particular, show how to use the universal property of presentations in order to prove that the above map defines a group homomorphism. The surjectivity is obvious, while the injectivity requires more arguments.)

In the sequel, we will prove that we may see $W(A_n)$ as a finite group of isometries of $\mathbb{R}^n$ generated by reflections.

3 Irreducibility

The idea of *irreducibility* is to define *elementary building blocks* for the the theory of Coxeter systems, so that an arbitrary Coxeter group can be build as a direct product of these elementary building blocks.

Definition 3.1 (*Irreducible Coxeter system*)

A Coxeter system (W, S) is called ***irreducible*** *if the corresponding Coxeter graph is connected. By abuse of language, we may also say that the Coxeter group W, or the Coxeter Graph, is irreducible.*

Lemma 3.2

Assume $G = G_1 \sqcup G_2$ is a disconnected Coxeter graph, where both G_1 and G_2 have non-empty vertex sets and no edge of G links a vertex of G_1 to a vertex of G_2. Then

$$W(G) \cong W(G_1) \times W(G_2)\,.$$

Proof: Exercise!

[Use the universal property of presentations, in order to define a homomorphism from $W(G)$ to $W(G_1) \times W(G_2)$. Prove that it is bijective. Emphasise why it is necessary that G_1 and G_2 are disjoint.] ■

Consequence 3.3

An induction argument shows that the Coxeter graph G associated to a Coxeter system (W, S) can be decomposed into connected components $G = \bigsqcup_{i=1}^{m} G_i$ such that

$$W(G) \cong W(G_1) \times \cdots \times W(G_m) .$$

It follows that to classify the Coxeter groups, it is enough to classify the irreducible ones.

4 First vision of the finite Coxeter groups

In this section, we take a first look at the cases, where W is irreducible and finite. We will prove later that the list below actually provides us with a complete classification of the finite Coxeter groups. Let n be the cardinality of the set S of generators.

The case $n = 1$

If $n = 1$, then the Coxeter graph is forced to be

$$A_1 \qquad \bullet$$

or in other words consists of a single vertex and no edges. Then $W(A_1) = \langle s_1 \mid s_1^2 = 1 \rangle \cong C_2$.

The case $n = 2$

If $n = 2$, then the Coxeter graph is

$$I_2(m) \qquad \bullet \overset{m}{\text{---}} \bullet \qquad (m \geqslant 3)$$

with $W(I_2(m)) = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1 s_2)^m = 1 \rangle \cong D_{2m}$, namely the dihedral group of order $2m$, which is the isometry group of the regular m-gone. (See Appendix B.) Notice that $m = 3$ gives again the graph A_2. The case $m = 4$ is rather known as B_2, and the case $m = 6$ as G_2.

The case $n = 3$

If $n = 3$, then there are 3 pairwise distinct Coxeter graphs corresponding to finite Coxeter groups:

$$A_3 \qquad \bullet \text{---} \bullet \text{---} \bullet \qquad \text{with } W(A_3) \cong \mathfrak{S}_4 ,$$

which is the isometry group of the regular 3-simplex;

$$B_3 \qquad \bullet \text{---} \bullet \overset{4}{\text{---}} \bullet \qquad \text{with } W(B_3) \cong C_2^3 \rtimes \mathfrak{S}_3 ,$$

which is the isometry group of the cube and of the octahedron; and

$$H_3 \qquad \bullet \text{---} \bullet \overset{5}{\text{---}} \bullet \qquad \text{with } W(H_3) \cong \mathfrak{A}_5 \times C_2 ,$$

which is the isometry group of the dodecahedron and of the icosahedron.

The case $n = 4$

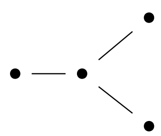
If $n = 4$, then there are 5 pairwise distinct Coxeter graphs corresponding to finite Coxeter groups:

$$A_4 \qquad \bullet - \bullet - \bullet - \bullet \qquad \text{with } W(A_4) \cong \mathfrak{S}_5\,,$$

which is the isometry group of the regular 4-simplex;

$$B_4 \qquad \bullet - \bullet - \bullet \overset{4}{-} \bullet \qquad \text{with } W(B_4) \cong C_2^4 \rtimes \mathfrak{S}_4\,,$$

which is the isometry group of the regular hypercube in $\mathbb{R}^4$;

D_4  with $W(D_4) \cong C_2^3 \rtimes \mathfrak{S}_4$

which does not correspond to any isometry group of a regular polytope;

$$F_4 \qquad \bullet - \bullet \overset{4}{-} \bullet - \bullet \qquad \text{with } W(F_4) \cong W(D_4) \rtimes \mathfrak{S}_3\,,$$

which is the isometry group of an exceptional regular polytope with 24 octahedral faces;

$$H_4 \qquad \bullet - \bullet - \bullet \overset{5}{-} \bullet \qquad \text{with } |W(H_4)| = 14400\,,$$

which is the isometry group of two regular polytopes (dual to each other) with 100 (resp. 600) dodecahedral (resp. tetrahedral) faces.

The case $n \geqslant 5$

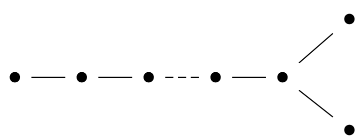
If $n \geqslant 5$, then the pairwise distinct Coxeter graphs corresponding to finite Coxeter groups are:

$$A_n \qquad \bullet - \bullet - \bullet \cdots \bullet - \bullet \qquad \text{with } W(A_n) \cong \mathfrak{S}_{n+1}\,,$$

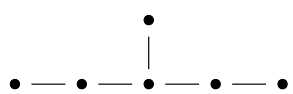
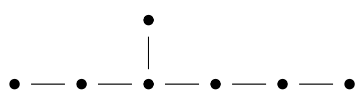
which is the isometry group of the regular n-simplex;

$$B_n = C_n \qquad \bullet - \bullet - \bullet \cdots \bullet \overset{4}{-} \bullet \qquad \text{with } W(B_n) \cong C_2^n \rtimes \mathfrak{S}_n\,,$$

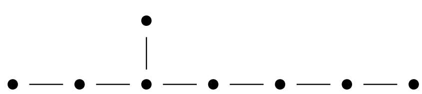
which is the isometry group of the regular hypercube in $\mathbb{R}^n$;

D_n  with $W(D_n) \cong C_2^{n-1} \rtimes \mathfrak{S}_n$;

and the three so-called *exceptional graphs*:

E_6  E_7 

and

E_8 

with $|W(E_6)| = 2^7 \cdot 3^4 \cdot 5$, $|W(E_7)| = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$, and $|W(E_8)| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$.

Chapter 2. Algebraic and Geometric Properties

The aim of this chapter is to study geometric properties of the Coxeter systems (W, S). Provided W is finite, in order to achieve this goal, we are going to *represent* W as a group generated by reflections w.r.t. hyperplanes in the n-dimensional euclidean space $\mathbb{R}^n$, where $n = |S|$. This will enable us to reduce the classification problem of the finite Coxeter groups to a problem of linear algebra over $\mathbb{R}$.

References:

[Hum90] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics Press, vol. 29, Cambridge University Press, Cambridge, 1990.

5 Deletion and exchange conditions

Throughout this section W is a group generated by a finite set $S \subset W\backslash\{1_W\}$ and we assume that $s^2 = 1$ for each $s \in S$.

Definition 5.1

Let $w \in W$.

(a) *An expression $w = s_1 \cdots s_r$ with $s_1, \ldots, s_r \in S$ is said to be* ***reduced*** *if any expression of w in the generators in S possesses at least r terms.*

(b) *The* ***length*** *of $w \neq 1$, denoted $\ell(w)$, is the number of terms in a reduced expression of w. By convention, $\ell(1_W) = 0$.*

Note: we need to prove that $\ell(w)$ is well-defined. Assuming it is, then we have the following properties:

Proposition 5.2 (*Elementary properties of the length*)

(1) $\ell(w) = 1 \iff w \in S$.

(2) $\ell(ww') \leqslant \ell(w) + \ell(w')$ *for every* $w, w' \in W$.

(3) $\ell(w^{-1}) = \ell(w)$ *for every* $w \in W$.

(4) *For each $s \in S$ and each $w \in W$, we have $\ell(sw) = \ell(w) \pm 1$.*

Proof: (1)–(3): exercise!

(4): Since the generators in S all have order 2 the map $\varepsilon : W \longrightarrow \{\pm 1\}, t \mapsto -1\ \forall\, t \in S$ defines a group homomorphism. Therefore:

- if $w \in W$ and $\ell(w)$ is even, then $\varepsilon(w) = 1$,
- if $w \in W$ and $\ell(w)$ is odd, then $\varepsilon(w) = -1$.

Clearly $\ell(sw) = \ell(w) + 1$ is a possible case and it is always true that $\ell(sw) \leqslant \ell(w) + 1$. Now, assume that $\ell(sw) < \ell(w) + 1$, then

$$\ell(sw) = \ell(w) - n$$

where $n \in \mathbb{Z}_{>0}$ is odd since

$$\varepsilon(sw) = \varepsilon(s)\varepsilon(w) = -\varepsilon(w)\,.$$

(In other words, the lengths of sw and w do not have the same parity.) Moreover,

$$\ell(w) = \ell(ssw) \leqslant \ell(sw) + 1 \quad \Longleftrightarrow \quad \ell(sw) \geqslant \ell(w) - 1\,.$$

In other words, if $\ell(sw) \neq \ell(w) + 1$, then $\ell(sw) = \ell(w) - 1$. ■

Notation: if $s_1 \cdots s_r$ is an expression in the generators $s_1, \ldots, s_r \in S$, then the notation

$$s_1 \cdots \check{s}_i \cdots s_r$$

means that s_i is deleted from this expression. In other words, $s_1 \cdots \check{s}_i \cdots s_r = s_1 \cdots s_{i-1}s_{i+1} \cdots s_r$.

Deletion Condition

We say that (W, S) satisfies the **deletion condition (DC)** if for any non-reduced expression $w = s_1 \cdots s_r$ with $s_1, \ldots, s_r \in S$, there exists $1 \leqslant i < j \leqslant r$ such that

$$w = s_1 \cdots \check{s}_i \cdots \check{s}_j \cdots s_r\,.$$

Exchange Condition

We say that (W, S) satisfies the **exchange condition (EC)** if for any reduced expression $w = s_1 \cdots s_r$ with $s_1, \ldots, s_r \in S$ and for any $s \in S$ such that $\ell(sw) \leqslant \ell(w)$, there exists $1 \leqslant j \leqslant r$ such that

$$w = ss_1 \cdots \check{s}_j \cdots s_r\,.$$

Proposition 5.3

Let (W, S) be as above. The deletion condition **(DC)** *and the exchange condition* **(EC)** *are equivalent.*

Proof:

"$\Rightarrow$" Assume **(DC)** holds. Let $w = s_1 \cdots s_r$ be a reduced expression and let $s \in S$ such that $\ell(sw) \leqslant \ell(w)$. Then

$$sw = ss_1 \cdots s_r$$

has $r + 1$ terms, hence is not reduced. Therefore **(DC)** implies that 2 letters can be deleted from this expression. We claim that one of these two letter must be s. Indeed, otherwise

$$sw = ss_1 \cdots \check{s}_i \cdots \check{s}_j \cdots s_r \;\Rightarrow\; w = s_1 \cdots \check{s}_i \cdots \check{s}_j \cdots s_r$$

which contradicts the fact that the length of w is r. Therefore,

$$sw = \check{s}s_1 \cdots \check{s}_j \cdots s_r \Rightarrow w = s^2 w = s(sw) = ss_1 \cdots \check{s}_j \cdots s_r .$$

"$\Leftarrow$" Assume now that **(EC)** holds and let $w = s_1 \cdots s_r$ with $s_1, \ldots, s_r \in S$ be a non-reduced expression. Let $i :=$ max s.t. $s_i s_{i+1} \cdots s_r$ is non-reduced ($1 \leqslant i \leqslant r-1$). Then for $s_i w' := s_i s_{i+1} \cdots s_r$, we have

(i) $\ell(s_i w') < r - i + 1$; and

(ii) $\ell(s_i w') \leqslant r - (i+1) + 1 = r - i = \ell(w')$.

Therefore **(EC)** implies that there exists an index j such that $i+1 \leqslant j \leqslant r$ and $w' = s_i s_{i+1} \cdots \check{s}_j \cdots s_r$. It follows that

$$w = s_1 \cdots s_i w' = s_1 \cdots s_i s_{i+1} \cdots \check{s}_j \cdots s_r$$

■

Theorem 5.4 (*Matsumoto*)

*The pair (W, S) is a Coxeter system $\Longleftrightarrow$ (W, S) satisfies **(DC)** $\Longleftrightarrow$ (W, S) satisfies **(EC)**.*

Proof: Without proof in this seminar. A proof can be found in [?]. ■

Theorem 5.5

*Let $W \leqslant O(n)$ be a finite group generated by a finite set S of orthogonal reflections of $\mathbb{R}^n$. Then (W, S) satisfies **(DC)**, hence is a Coxeter system.*

Proof: Without proof in this seminar. A proof can be found in [?]. ■

Remark 5.6

Theorem 5.5 actually provides us with a method to obtain all the finite Coxeter systems listed in Chapter 1.

6 Informal example: the dihedral groups

[At this stage I will give an informal example on the board about the underlying geometry of the dihedral groups.]

7 Geometry and representations

From now on, we let (W, S) be a Coxeter system with $S = \{s_1, \ldots, s_n\}$ and V be an n-dimensional $\mathbb{R}$-vector space with ordered basis $(e_1, \ldots, e_n)$.

Definition 7.1 (*Canonical bilinear form, reflections and hyperplanes*)

(1) *The **canonical bilinear form** associated to (W, S) is the $\mathbb{R}$-bilinear form defined by*

$$\begin{array}{rccl} B: & V \times V & \longrightarrow & \mathbb{R} \\ & (e_i, e_j) & \mapsto & B(e_i, e_j) := -\cos \frac{\pi}{m_{ij}} . \end{array}$$

(2) *For $1 \leqslant i \leqslant n$, the **reflection** associated to e_i and B is the reflection*

$$\begin{array}{rccl} \sigma_i : & V & \longrightarrow & V \\ & x & \mapsto & x - 2B(e_i, x)e_i \end{array}$$

*and the **hyperplane** associated to e_i and B is $H_i := \ker B(-, e_i) = \{x \in V \mid B(x, e_i) = 0\}$.*

Remark 7.2 (***Properties of B and σ_i***)

(1) · $m_{ii} = 1 \Longrightarrow B(e_i, e_i) = -\cos\pi = 1$
· $m_{ij} = 2 \Longrightarrow B(e_i, e_j) = -\cos\frac{\pi}{2} = 0$
· $m_{ij} = \infty \Longrightarrow B(e_i, e_j) = -\cos 0 = -1$

(2) *The form B is symmetric since $m_{ij} = m_{ji}$ for all $1 \leqslant i, j \leqslant n$.*

(3) *Warning: B is not necessarily positive definite, so that B need not be a scalar product in general.*

(4) *The reflection σ_i has order 2. Indeed, for all $x \in V$, we have:*

$$\begin{aligned} \sigma_i \circ \sigma_i(x) &= \sigma_i\left(x - 2B(e_i, x)e_i\right) = x - 2B(e_i, x)e_i - 2B(e_i, x - 2B(e_i, x)e_i)e_i \\ &= x - 2B(e_i, x)e_i - 2\left[B(e_i, x) - 2B(e_i, x)\underbrace{B(e_i, e_i)}_{=1}\right]e_i \\ &= x - 2B(e_i, x)e_i - 2\left[-B(e_i, x)\right]e_i \\ &= x \end{aligned}$$

Hence $\sigma_i \circ \sigma_i$ is the identity map.

(5) *The map $B(e_i, -)$ is a non-zero $\mathbb{R}$-linear form, so that its image is the whole of $\mathbb{R}$. Therefore, it follows from the Rank–nullity theorem that*

$$\dim_{\mathbb{R}} H_i = n - \dim_{\mathbb{R}}(Im B(e_i, -)) = n - 1\,.$$

(6) *We have:*

· $\sigma_i(x) = x \Longleftrightarrow B(e_i, x) = 0 \Longleftrightarrow x \in H_i$, *and*
· $\sigma_i(e_i) = e_i - 2 \cdot 1 \cdot e_i = -e_i$.

Therefore σ_i is indeed a reflection of Hyperplane H_i.

Lemma 7.3

For each $1 \leqslant i \leqslant n$ the reflection σ_i is an $\mathbb{R}$-linear transformation which is orthogonal with respect to B. (One also says that the σ_i's preserve B.)

Proof: The $\mathbb{R}$-linearity is clear by definition. We only prove that σ_i ($1 \leqslant i \leqslant n$) is orthogonal with respect to B. Notice that each $x, y \in \mathbb{R}^n$ may be written as

$$x = u + \lambda e_i \qquad \text{and} \qquad y = v + \mu e_i \qquad \text{with} \qquad u, v \in H_i, \lambda, \mu \in \mathbb{R}$$

since H_i is a hyperplane and $e_i \notin H_i$. Therefore,

$$\sigma_i(x) = u - \lambda e_i \qquad \text{and} \qquad \sigma_i(y) = v - \mu e_i$$

by Remark 7.2(6) and it follows from the $\mathbb{R}$-bilinearity of B that

$$\begin{aligned} B(\sigma_i(x), \sigma_i(y)) = B(u - \lambda e_i, v - \mu e_i) &= B(u, v) - \lambda \underbrace{B(e_i, v)}_{=0} - \mu \underbrace{B(u, e_i)}_{=0} + \lambda\mu B(e_i, e_i) \\ &= B(u, v) + \lambda \underbrace{B(e_i, v)}_{=0} + \mu \underbrace{B(u, e_i)}_{=0} + \lambda\mu B(e_i, e_i) \\ &= B(u + \lambda e_i, v + \mu e_i) \\ &= B(x, y)\,, \end{aligned}$$

as required. ■

Theorem 7.4

(a) *The map defined by*

$$\begin{array}{rccc} \sigma : & W & \longrightarrow & \mathrm{GL}(V) \\ & s_i & \mapsto & \sigma_i \end{array}$$

is a group homomorphism, called the ***canonical representation associated to*** W.

(b) $\mathrm{Im}(\sigma) \subseteq \mathcal{O}(V, B) := \{\varphi \in \mathrm{GL}(V) \mid \varphi \text{ preserves } B\}$.

(c) *The integer m_{ij} is the order of $s_i s_j$ in W for all $1 \leqslant i < j \leqslant n$.*

Proof: (a) By the universal property of presentations (B.6 of the Appendix), it suffices to check that the relations defining W are mapped to the identity map on V by σ.

- For the relations $s_i^2 = 1$ ($1 \leqslant i \leqslant n$), it is obvious since we have seen in 7.2(4) that σ_i^2 has order 2.
- For the relations $(s_i s_j)^{m_{ij}} = 1$ with $i \neq j$ and $2 \leqslant m_{ij} < \infty$, we consider the plane $P := \mathbb{R}e_i \oplus \mathbb{R}e_j$ in $\mathbb{R}^n$. Then the matrix of $B|_P : P \times P \longrightarrow \mathbb{R}$ w.r.t. the basis (e_i, e_j) is

$$\begin{pmatrix} 1 & -\cos\frac{\pi}{m_{ij}} \\ -\cos\frac{\pi}{m_{ij}} & 1 \end{pmatrix}$$

and we compute

$$\begin{aligned} B(\lambda e_i + \mu e_j) &= \lambda^2 \underbrace{B(e_i, e_i)}_{=1} + 2\lambda\mu B(e_i, e_j) + \mu^2 \underbrace{B(e_j, e_j)}_{=1} \\ &= \lambda^2 + 2\lambda\mu(-\cos\frac{\pi}{m_{ij}}) + \mu^2 \\ &= \lambda^2 + 2\lambda\mu(-\cos\frac{\pi}{m_{ij}}) + \mu^2(\cos^2\frac{\pi}{m_{ij}} + \sin^2\frac{\pi}{m_{ij}}) \\ &= \left(\lambda - \mu\cos\frac{\pi}{m_{ij}}\right)^2 + \left(\mu\sin\frac{\pi}{m_{ij}}\right)^2 \geqslant 0\,. \end{aligned}$$

Therefore $B|_P$ is positive definite and $V = P \oplus Q$ with $Q = P^\perp$ the orthogonal subspace to P w.r.t. to B. (Notice that $B|_P$ is also non-degenerate, since otherwise there would be a

$0 \neq v \in P$ such that $B|_P(v, w) = 0$ for all $w \in P$ and in particular we would have $B|_P(v, v) = 0$, which contradicts the fact that $B|_P$ is positive definite.)
It follows that $H_i = e_i^\perp \supset P^\perp = Q$ and similarly $H_j \supset Q$. Thus $\sigma_i|_Q = \mathrm{Id}$ and $\sigma_j|_Q = \mathrm{Id}$, so that both σ_i and σ_j are entirely characterised by their restriction to P. In particular, whether the relation $(\sigma_i\sigma_j)^{m_{ij}} = \mathrm{Id}$ holds can be tested on P. In fact, because P together with $B|_P$ can be identified with the euclidean space $\mathbb{R}^2$ with its standard scalar product, by Theorem B.7 and its proof, we have that $(\sigma_i\sigma_j)|_P$ is a rotation of angle $\frac{2\pi}{m_{ij}}$ in $P \cong \mathbb{R}^2$ and hence $(\sigma_i\sigma_j)^{m_{ij}} = \mathrm{Id}$.

(b) By Lemma 7.3, $\sigma_i \in \mathcal{O}(V, B)$ for each $1 \leqslant i \leqslant n$, whence $\mathrm{Im}(\sigma) \subseteq \mathcal{O}(V, B)$.

(c) We differentiate between two cases:

(i) $m_{ij} < \infty$: By (a), m_{ij} is the order of $(\sigma_i\sigma_j)|_P$ in P, that is $((\sigma_i\sigma_j)|_P)^m \neq \mathrm{Id}$ if $1 \leqslant m < m_{ij}$). But as σ is a group homomorphism, we must also have that $(s_is_j)^m \neq \mathrm{Id}$ if $1 \leqslant m < m_{ij}$, hence m_{ij} is the order of s_is_j.

(ii) $m_{ij} = \infty$: By definition, we have

$$\sigma_i(e_j) = e_j - 2B(e_i, e_j)e_i = e_j + 2e_i \qquad \text{and} \qquad \sigma_j(e_i) = e_i - 2B(e_j, e_i)e_j = e_i + 2e_j\,.$$

Hence $\sigma_i(e_i + e_j) = \sigma_j(e_i + e_j) = e_i + e_j$ and $\sigma_i\sigma_j(e_i + e_j) = e_i + e_j$. It follows that

$$\sigma_i\sigma_j(e_i) = \sigma_(e_i + 2e_j) = \sigma_(e_i + e_j) + \sigma_(e_j) = e_i + e_j + e_j + 2e_i = 2(e_i + e_j) + e_i$$

and an induction yields

$$(\sigma_i\sigma_j)^k(e_i) = 2k(e_i + e_j) + e_i \qquad \forall\, k \geqslant 1\,.$$

In particular, $(\sigma_i\sigma_j)^k(e_i) \neq e_i \;\; \forall\, k \geqslant 1$, so that we must have that the order of $\sigma_i\sigma_j$ is infinite, and therefore so is the order of s_is_j since σ is a group homomorphism. ∎

8 The dual representation

Let now $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ be the $\mathbb{R}$-dual of V and let $(b_1, \ldots, b_n)$ denote the dual basis to the basis $(e_1, \ldots, e_n)$ of V. Recall from linear algebra that any endomorphism $\alpha \in \mathrm{End}_{\mathbb{R}}(V) = \mathrm{Hom}_{\mathbb{R}}(V, V)$ induces an $\mathbb{R}$-linear endomorphism

$$\begin{array}{rccc} {}^t\alpha: & V^* & \longrightarrow & V^* \\ & f & \mapsto & {}^t\alpha(f) := f \circ \alpha \end{array}$$

and the matrix of ${}^t\alpha$ w.r.t. the basis $(b_1, \ldots, b_n)$ is the transpose of the matrix of α w.r.t. the basis $(e_1, \ldots, e_n)$.
Define

$$\begin{array}{rccc} \sigma^*: & W & \longrightarrow & \mathrm{GL}(V^*) \\ & w & \mapsto & \sigma^*(w) := {}^t(\sigma(w^{-1}))\,. \end{array}$$

Lemma 8.1

*The map σ^* is a group homomorphism, called the **dual representation** (to σ).*

Proof: Let $u, w \in W$. Then

$$\sigma^*(uw) = {}^t(\sigma((uw)^{-1})) = {}^t(\sigma(w^{-1}) \circ \sigma(u^{-1})) = {}^t(\sigma(u^{-1})) \circ {}^t(\sigma(w^{-1})) = \sigma^*(u) \circ \sigma^*(w)\,.$$

∎

Furthermore, let us denote the evaluation of $f \in V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ in $v \in V$ as follows:

$$\begin{array}{rccc} \langle -,- \rangle : & V^* \times V & \longrightarrow & V \\ & (f, v) & \mapsto & \langle f, v \rangle := f(v) \end{array}$$

and moreover, given $w \in W$, $v \in V$ and $f \in V^*$, we set $w.v := \sigma(w)(v)$ and $w.f := \sigma^*(w)(f)$.

Lemma 8.2

We have $\langle w.f, v \rangle = \langle f, w^{-1}.v \rangle \;\; \forall\, w \in W, \forall\, v \in V$ and $\forall\, f \in V^*$.

Proof:

$$\begin{aligned} \langle w.f, v \rangle = (w.f)(v) = (\sigma^*(w)(f))\,(v) = \left({}^t(\sigma(w^{-1}))(f)\right)(v) &= \left(f \circ \sigma(w^{-1})\right)(v) \\ &= f\left(\sigma(w^{-1})(v)\right) = \langle f, w^{-1}.v \rangle . \end{aligned}$$

■

9 Half-spaces and chambers

Given $1 \leqslant i \leqslant n$, we set

$$\mathsf{H}_i := \ker(\langle -, e_i \rangle) = \{f \in V^* \mid f(e_i) = 0\}\,,$$

which obviously admits the basis $(b_1, \ldots, \check{b}_i, \ldots, b_n)$. Moreover, we let

$$D_+(\mathsf{H}_i) := \{f \in V^* \mid \langle f, e_i \rangle > 0\} \quad \text{and} \quad D_-(\mathsf{H}_i) := \{f \in V^* \mid \langle f, e_i \rangle < 0\}\,.$$

Definition 9.1

(a) *The subset $C := \{f \in V^* \mid \langle f, e_i \rangle > 0 \;\; \forall\, 1 \leqslant i \leqslant n\} = \bigcap_{i=1}^n D_+(\mathsf{H}_i)$ of V^* is called the **fundamental chamber** of of V^*.*

(b) *The subsets $w.C := \{w.f \mid f \in C\}$ of V^* are called the **chambers** of V^*.*

Lemma 9.2

For each $s_i \in S$ the operation $s_i.f$ for f running through V^* is a reflection of hyperplane H_i which exchanges $D_+(\mathsf{H}_i)$ and $D_-(\mathsf{H}_i)$.

Proof: To begin with, for each $1 \leqslant j \neq i \leqslant n$ and every $v \in V$ we have:

$$\begin{aligned} \langle s_i.b_j, v \rangle \overset{Lem.8.2}{=} \langle b_j, s_i.v \rangle = \langle b_j, v - 2B(v, e_i)e_i \rangle &= \langle b_j, v \rangle - 2B(v, e_i) \overbrace{\langle b_j, e_i \rangle}^{\delta_{ji}} \\ &= \langle b_j, v \rangle \end{aligned}$$

Hence $s_i.b_j = b_j$, which proves that the map $s_i.(-)$ is the identity on the hyperplane H_i. Furthermore, if $i = j$, then the above calculation yields

$$\langle s_i.b_i, v \rangle = \langle b_i, v \rangle - 2B(v, e_i)\,,$$

so that $s_i.b_i = b_i - 2B(-, e_i)$. Now, on the one hand $b_i \in D_+(\mathsf{H}_i)$ since $\langle b_i, e_i\rangle = 1 > 0$, and on the other hand, $s_i.b_i \in D_-(\mathsf{H}_i)$ since

$$\langle s_i.b_i, e_i\rangle = \langle b_i, e_i\rangle - 2B(e_i, e_i) = 1 - 2\cdot 1 = -1 < 0\,.$$

It follows that $s_i.f \in D_-(\mathsf{H}_i)$ for every $f \in D_+(\mathsf{H}_i)$, and conversely $s_i.f \in D_+(\mathsf{H}_i)$ for every $f \in D_-(\mathsf{H}_i)$, as required. ■

Exercise 9.3

Let $w \in W$ and $s_i \in S$. Then,

$$\begin{cases} w.C \subseteq D_+(\mathsf{H}_i) & \Longleftrightarrow \quad \ell(s_i w) = \ell(w) + 1\,; \text{ and} \\ w.C \subseteq D_-(\mathsf{H}_i) & \Longleftrightarrow \quad \ell(s_i w) = \ell(w) - 1\,. \end{cases}$$

Proof: Please write the solution on your own. ■

Theorem 9.4 (*Tits*)

Let C be the fundamental chamber in V^*. Then

$$w.C \cap C = \varnothing \qquad \forall\, w \in W\setminus\{1\}\,.$$

Proof: Let $w \in W\setminus\{1\}$ and let $w = t_1 \cdots t_r$ with $t_1, \dots, t_r \in S$ be a reduced expression for w. Then $\ell(t_1 w) = \ell(w) - 1$ and it follows from Exercise 9.3 that

$$w.C \subseteq D_-(\mathsf{H}_1) \qquad \text{and} \qquad C \subseteq D_+(\mathsf{H}_1)\,.$$

Hence, $w.C \cap C \subseteq D_-(\mathsf{H}_1) \cap D_+(\mathsf{H}_1) = \varnothing$ by definition. ■

Fundamental Corollary 9.5

Both $\sigma : W \longrightarrow \mathrm{GL}(V)$ and $\sigma^* : W \longrightarrow \mathrm{GL}(V^*)$ are injective. In particular, W, $\sigma(W)$ and $\sigma^*(W)$ are isomorphic groups.

Proof: We need to prove that the kernels of σ and σ^* are trivial. So, let $w \in W$.

- To begin with, $\sigma^*(w) = \mathrm{Id}_{V^*} \Longrightarrow w.f = \sigma^*(w)(f) = f$ for every $f \in V^*$, so that $w.C = C$ by definition and it follows from the theorem of Tits that $w = 1$. Hence $\ker(\sigma^*) = \{1\}$.
- Next, we use the fact that $\sigma^*(w) = {}^t\sigma(w^{-1})$. It follows that:

$$\sigma(w) = \mathrm{Id}_V \quad\Longrightarrow\quad \sigma(w^{-1}) = \mathrm{Id}_V^{-1} = \mathrm{Id}_V \quad\Longrightarrow\quad {}^t\sigma(w^{-1}) = {}^t\mathrm{Id}_V = \mathrm{Id}_{V^*}\,.$$

Therefore $\ker(\sigma^*) = \{1\} \Longrightarrow \ker(\sigma) = \{1\}$ as well. ■

Proposition 9.6 (*Requires* Einführung in die Topologie)

The subgroup $\sigma(W)$ of $\mathrm{GL}(V)$ is closed *and* discrete *— where $\mathrm{GL}(V)$ is seen as topological subspace of $M_n(\mathbb{R}) \simeq (\mathbb{R}^{n^2}$, standard topology$)$ and endowed with the induced topology.*

Proof: Accepted without proof. ■

10 Irreducibility of representations

For the terminology used in this section, we refer to Appendix C and we note that the canonical bilinear form B is W-invariant by Theorem 7.4(b).

Proposition 10.1

If (W, S) is an irreducible Coxeter system, then the following holds:

(a) *Any proper W-invariant subspace of V is contained in* $\ker B$*.*

(b) *$w.u = u$ for every $w \in W$ and every $u \in \ker B$, so that in particular $\ker B$ is W-invariant.*

Proof: Set $U := \ker B = \{x \in V \mid B(x, y) = 0\ \forall y \in V\}$.

(a) Let $V' \subsetneq V$ be a W-invariant subspace of V. We treat two cases:

Case 1: $\exists$ an index i such that $e_i \in V'$. Let $j \neq i$ such that $m_{ij} \geqslant 3$, so that $\cos \frac{\pi}{m_{ij}} > 0$. Thus,

$$V' \ni s_j.e_i = e_i - 2B(e_i, e_j)e_j = e_i - 2(-\cos \frac{\pi}{m_{ij}})e_j$$

and

$$0 \neq 2B(e_i, e_j)e_j = e_i - s_j.e_i \in V' \quad \Longrightarrow \quad e_j \in V' .$$

Since (W, S) is irreducible, its Coxeter graph is connected and the above argument proves that $e_k \in V'$ for every $1 \leqslant k \leqslant n$, i.e. $V' = V$, which is a contradiction.

Case 2: $e_i \neq V'$ for every $1 \leqslant i \leqslant n$. Now if $v' \in V'$, then

$$s_i.v' = v' - 2B(e_i, v')e_i \quad \Longrightarrow \quad 2B(e_i, v')e_i = \underbrace{v'}_{\in V'} - \underbrace{s_i.v'}_{\in V'} \in V'$$

and since $e_i \neq V'$, we must have $B(e_i, v') = 0$. Hence $v' \in U$ and $V' \subseteq U$.

(b) For each $1 \leqslant i \leqslant n$ and each $u \in U$ holds $B(e_i, u)$, so that

$$s_i.u = \sigma(s_i)(u) = u - 2\underbrace{B(e_i, u)}_{=0} e_i = u .$$

As W is generated by S, it follows that $w.u = u$ for every $w \in W$ and $u \in U$. ■

Theorem 10.2

Let (W, S) be an irreducible Coxeter system. Then:

$$\sigma \text{ is irreducible} \quad \Longleftrightarrow \quad B \text{ is non-degenerate}$$

In which case σ is in fact absolutely irreducible.

Proof: Propostion 10.1 implies that any W-invariant proper subspace of V is contained in $U := \ker B$. Therefore, we have the following equivalences:

$$\begin{aligned} \sigma \text{ is irreducible} \quad &\Longleftrightarrow \quad \text{there is no proper } W\text{-invariant subspace of } V \\ &\overset{\textit{Prop. } 10.1}{\Longleftrightarrow} \quad U = \{0\} \\ &\overset{\textit{Definition}}{\Longleftrightarrow} \quad B \text{ is non-degenerate} \end{aligned}$$

Now we claim that σ irreducible $\Longrightarrow$ σ absolutely irreducible.

Let $1 \leqslant i \leqslant n$. Then $\sigma(s_i) =: \sigma_i$ is a reflection of V with fixed hyperplane $H_i = \{v \in V \mid B(e_i, v) = 0\}$. Let α be an endomorphism of σ. Then,

$$\alpha \circ (\sigma_i - \mathrm{Id}_V)(v) = \alpha(s_i.v - v) = \alpha(s_i.v) - \alpha(v) = s_i.\alpha(v) - \alpha(v) = (\sigma_i - \mathrm{Id}_V) \circ \alpha(v) \qquad \forall\, v \in V\,.$$

Hence $\alpha \circ (\sigma_i - \mathrm{Id}_V) = (\sigma_i - \mathrm{Id}_V) \circ \alpha$.
Moreover,

$$(\sigma_i - \mathrm{Id}_V)(v) = (\sigma(s_i) - \mathrm{Id}_V)(v) = v - 2B(e_i, v)e_i - v = -2B(e_i, v)e_i$$

for every $v \in V$, so that $\mathrm{Im}(\sigma_i - \mathrm{Id}_V) = \mathbb{R}e_i$. Hence by the above $\alpha(\mathbb{R}e_i) \subseteq \mathbb{R}e_i$, and therefore there exists $\lambda \in \mathbb{R}$ such that $\alpha(e_i) = \lambda e_i$. Then $V' := \{v \in V \mid \alpha(v) = \lambda \cdot v\}$ is by construction a W-invariant subspace of V containing $\mathbb{R}e_i$ (hence non-zero) since:

$$\forall\, w \in W, \forall\, v' \in V', \alpha(w.v') = w.\alpha(v') = w.(\lambda \cdot v') = \lambda \cdot (w.v') \;\Rightarrow\; w.v' \in V'$$

Therefore, as we assume that σ is irreducible, we must have $V \neq V'$ and it follows that $\alpha = \lambda \cdot \mathrm{Id}_V$, as required. ■

Chapter 3. Classification of the finite Coxeter groups

The aim of this chapter is now to classify the finite Coxeter groups using linear algebra and graph theory. First we see that the finiteness of a Coxeter group is equivalent to the fact that the associated canonical bilinear form is positive definite. Second we use this fact to provide a constructive proof of all possible finite Coxeter groups as we already described them in Chapter 1.

References:

[Hum90] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics Press, vol. 29, Cambridge University Press, Cambridge, 1990.

11 The finiteness theorem

Theorem 11.1 (*Finiteness Theorem*)

Let (W, S) be an irreducible Coxeter system. Then W is finite if and only if the associated canonical $\mathbb{R}$-bilinear form B is positive definite.

Proof:

"$\Rightarrow$" Assume that W is a finite group. Let $U := \ker B$. Clearly $U \subsetneq V$ since e.g. $B(e_1, e_1) = 1 \neq 0$. Thus by Maschke's Theorem (see Appendix C) there exists a W-invariant subspace $U' \subseteq V$ such that $V = U \oplus U'$. However, by Proposition 10.1(a), if U' is a W-invariant subspace of V, then either $U' \subseteq U$ or $U' = V$. Hence $U' = V$ and it follows immediately that $U = \{0\}$, so that B is non-degenerate. It now follows from Theorem 10.2 that σ is absolutely irreducible.
Now, we may consider the standard scalar product $\langle -, - \rangle_V$ on V. It is then easily checked that

$$A : V \times V \longrightarrow \mathbb{R}, (v, v') \mapsto A(v, v') := \sum_{w \in W} \langle w.v, w.v' \rangle_V$$

is an $\mathbb{R}$-bilinear form, which is W-invariant (Exercise!) and positive definite (since $\langle -, - \rangle_V$ is). By Proposition C.2(b), there exists $\lambda \in \mathbb{R}$ such that $B = \lambda A$. In particular

$$1 = B(e_1, e_1) = \lambda \underbrace{A(e_1, e_1)}_{>0} \qquad \Longrightarrow \qquad \lambda > 0 .$$

It now follows that B is positive definite since A is.

"$\Leftarrow$" We omit this part of the proof as it requires arguments using the *Einführung in die Topologie*.

■

Terminology: If (W, S) is a Coxeter system and Γ_W is the associated Coxeter graph, then by abuse of language we say that Γ_W is **positive definite** if the associated canonical bilinear form B is positive definite. We also write $\det(\Gamma_W)$ instead of $\det(B)$.

12 The classification

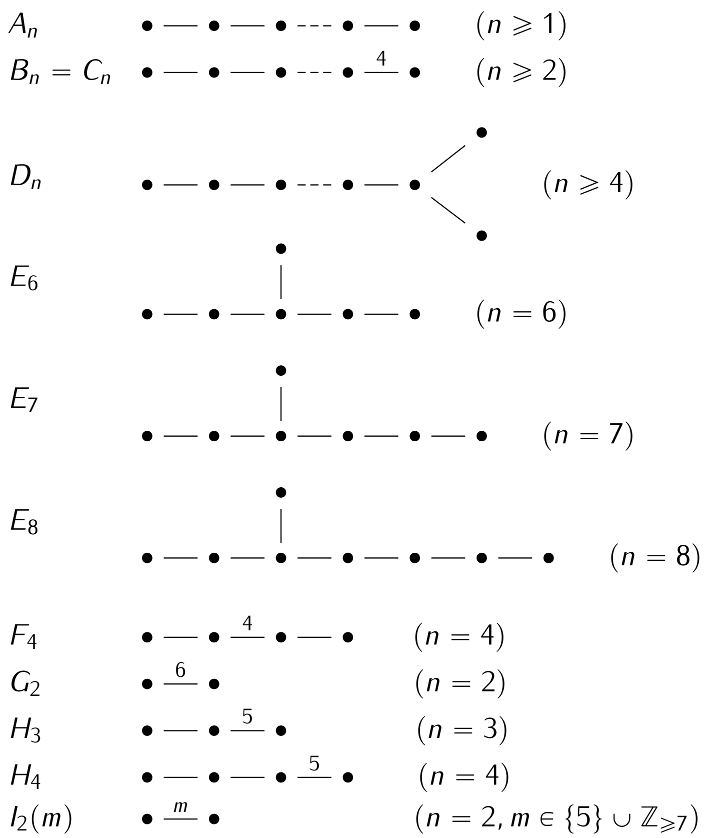
As we have seen in Chapter 1, in order to classify the Coxeter systems, and hence the Coxeter groups, it is enough to classify the *irreducible* Coxeter systems, in which case the associated Coxeter graph Γ_W is connected.

Moreover, by the Finiteness Theorem, finding the irreducible Coxeter systems (W, S) such that W is *finite* is equivalent to finding the positive definite associated canonical $\mathbb{R}$-bilinear forms B, the definition of which depends only Γ_W. Hence we are reduced to the following graph theory problem:

> Which are the connected Coxeter graphs Γ_W for which the associated canonical $\mathbb{R}$-bilinear form is positive definite?

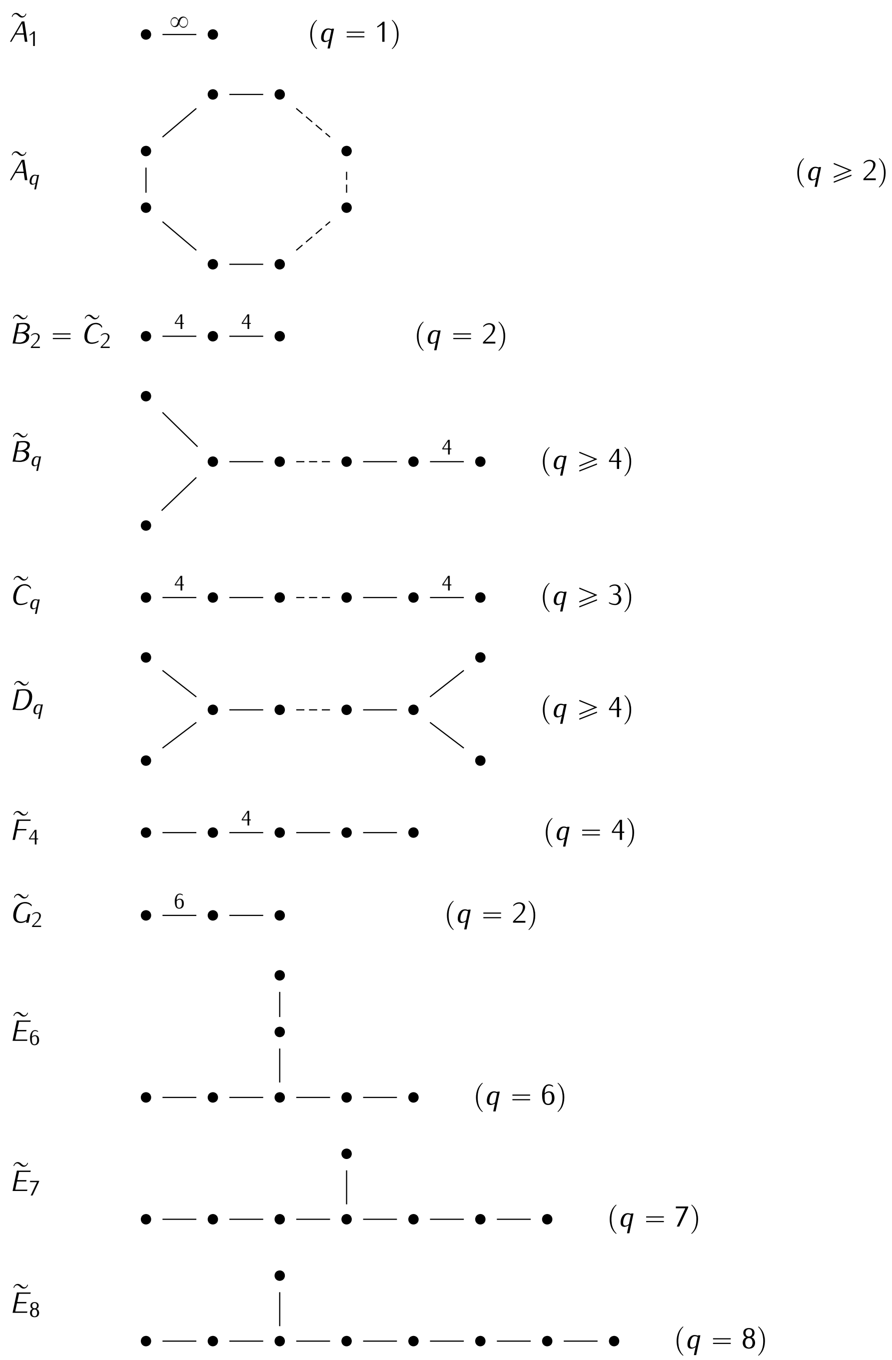
Theorem A (*Positive definite Coxeter graphs*)

*Let Γ_W be an irreducible Coxeter graph with $n \in \mathbb{Z}_{>0}$ vertices. Then Γ_W is positive definite if and only if Γ_W belongs to the following list (*List A*):*



Theorem B (*Positive semi-definite Coxeter graphs*)

*Let Γ_W be an irreducible Coxeter graph with $q+1$ vertices ($q \in \mathbb{Z}_{>0}$). Then Γ_W is positive semi-definite if and only if Γ_W belongs to the following list (*List B*):*



Furthermore, if Γ_W is positive semi-definite but not positive definite, then $\dim_{\mathbb{R}}(\ker B) = 1$.

We are going to prove Theorem A and Theorem B together in two seperate proofs, the first one dealing with the sufficient condition (i.e. the direction "$\Leftarrow$") and the second one dealing with the necessary condition (i.e. the direction "$\Rightarrow$").

For the sufficient condition, we need the following standard Criterion from linear algebra, which we accept here without proof:

Criterion for positive (semi-)definiteness of a symmetric $\mathbb{R}$-bilinear form

Let B be a symmetric $\mathbb{R}$-bilinear form on an $\mathbb{R}$-vector space of dimension $n \in \mathbb{Z}_{>0}$ with matrix $\mathrm{Mat}(B)$. For each $1 \leqslant i \leqslant n$, let B_i denote the principal minor of $\mathrm{Mat}(B)$ of size i. Then:

(a) B is positive definite $\iff \det(B_i) > 0$ for each $1 \leqslant i \leqslant n$.

(b) B is positive semi-definite with $\dim_{\mathbb{R}}(\ker B) = 1 \Longleftarrow \det(B_i) > 0$ for each $1 \leqslant i \leqslant n-1$ and $\det(B) = 0$.

Proof of Theorem A and Theorem B: sufficient condition "$\Leftarrow$":
We need to prove that:

$$\begin{aligned} \Gamma_W \in \text{List A} &\implies \Gamma_W \text{ is positive definite; and} \\ \Gamma_W \in \text{List B} &\implies \Gamma_W \text{ is positive semi-definite.} \end{aligned}$$

We proceed by induction on the number n of vertices of the graph Γ_W. Denote by $\mathrm{Mat}(B)$ the matrix of B w.r.t. the ordered basis $(e_1, \dots, e_n)$.

- $\underline{n=1}$: A_1 $\quad\bullet\quad$ yields $\mathrm{Mat}(B) = (1)$, hence $\det(B) = 1 > 0$.
- $\underline{n=2}$: A_2 $\quad\bullet\!-\!\!-\!\bullet\quad$ yields $\mathrm{Mat}(B) = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$, hence $\det(B) = \frac{3}{4} > 0$.
  - B_2 $\quad\bullet\overset{4}{-\!\!-}\bullet\quad$ yields $\mathrm{Mat}(B) = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 1 \end{pmatrix}$, hence $\det(B) = \frac{1}{2} > 0$.
  - G_2 $\quad\bullet\overset{6}{-\!\!-}\bullet\quad$ yields $\mathrm{Mat}(B) = \begin{pmatrix} 1 & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 1 \end{pmatrix}$, hence $\det(B) = \frac{1}{4} > 0$.
  - $I_2(m)$ $\quad\bullet\overset{m}{-\!\!-}\bullet\quad$ yields $\mathrm{Mat}(B) = \begin{pmatrix} 1 & -\cos\frac{\pi}{m} \\ -\cos\frac{\pi}{m} & 1 \end{pmatrix}$, hence $\det(B) = 1 - \cos^2(\frac{\pi}{m}) > 0$.
  - $\widetilde{A}_1$ $\quad\bullet\overset{\infty}{-\!\!-}\bullet\quad$ yields $\mathrm{Mat}(B) = \begin{pmatrix} 1 & -\cos\frac{\pi}{\infty} \\ -\cos\frac{\pi}{\infty} & 1 \end{pmatrix}$, hence $\det(B) = 0$.

Hence for $n = 1, 2$ all the graphs in List A and List B satisfy the above criterion. We may now assume that

- $\underline{n \geqslant 3}$: let Γ_W be in List A or List B with $n \geqslant 3$ vertices. We now remove an end vertex of Γ_W, apart for $\widetilde{A}_n$ for which we may remove an arbitrary vertex. We denote by Γ'_W the resulting graph and we observe that Γ'_W is in List A. Therefore by the induction hypothesis the matrix B' of Γ'_W is positive definite. Using the Criterion, it suffices to prove that $\det(\mathrm{Mat}(B)) > 0$ if $\Gamma_W \in$ List A and $\det(\mathrm{Mat}(B)) = 0$ if $\Gamma_W \in$ List B, where

$$\mathrm{Mat}(B) = \left(\begin{array}{c|c} B' & * \\ \hline * & 1 \end{array}\right).$$

A straightforward computation (Exercise!) yields:

$$\begin{aligned}
&\det(A_n) = \tfrac{n+1}{2^n} > 0 \quad (n \geqslant 3), && \det(F_4) = \tfrac{1}{2^4} > 0\,, \\
&\det(B_n) = \tfrac{1}{2^{n-1}} > 0 \quad (n \geqslant 3), && \det(H_3) = \tfrac{3-\sqrt{5}}{8} > 0\,, \\
&\det(D_n) = \tfrac{1}{2^{n-2}} > 0 \quad (n \geqslant 4), && \det(H_4) = \tfrac{7-3\sqrt{5}}{32} > 0\,, \\
&\det(E_n) = \tfrac{9-n}{2^n} > 0 \quad (\text{for } n = 7, 8, 9), && \det(\widetilde{X}) = 0 \text{ for } \widetilde{X} \text{ in List B}\,.
\end{aligned}$$

■

For the necessary condition, we will need to consider the subgraphs of Γ_W. We recall the following notion from graph Theory:

Definition 12.1 (*Subgraph*)

*Let Γ_W be an irreducible Coxeter graph. We call **subgraph** of Γ_W a graph Γ'_W formed from a subset of the vertices and edges of Γ_W, where the weight of an edge of Γ'_W is less or equal to the weight of the same edge seen as an edge of Γ_W.*

Proposition 12.2

Let Γ_W be an irreducible Coxeter graph with n vertices ($n \in \mathbb{Z}_{>0}$), which is either positive definite or positive semi-definite. Then any proper subgraph of Γ_W is positive definite.

Proof: Let Γ'_W be a proper subgraph of Γ_W. W.l.o.g. we may assume that the vertices of Γ_W are labelled such that the vertices of Γ'_W are $s_1, \ldots, s_m$ with $m \leqslant n$.
Write m'_{ij} for the weight of the edge (s_i, s_j) in Γ'_W. Let $\mathrm{Mat}(B) = (b_{ij})$ be the matrix of the canonical bilinear form B associated with Γ_W and $\mathrm{Mat}(B') = (b'_{ij})$ be the matrix of the canonical bilinear form B' associated with Γ'_W.
Clearly:

$$m'_{ij} \leqslant m_{ij} \quad \Longrightarrow \quad b'_{ij} = -\cos\frac{\pi}{m'_{ij}} \geqslant -\cos\frac{\pi}{m_{ij}} = b_{ij}$$

Assume now that Γ'_W is not positive definite. Thus there exists $0 \neq v \in V' := \langle e_1, \ldots, e_m \rangle_{\mathbb{R}}$ such that $B'(v,v) \leqslant 0$. Write v as a linear combination $v = \sum_{i=1}^n x_i e_i \in V'$ with $x_i \in \mathbb{R}$ for each $1 \leqslant i \leqslant n$ and $x_i = 0$ for each $m+1 \leqslant i \leqslant n$. Then for $|v| := \sum_{i=1}^n |x_i| e_i$, we have:

$$\begin{aligned} 0 \leqslant B(|v|,|v|) = \sum_{i,j=1}^{n} b_{ij}|x_i|\,|x_j| &\leqslant \sum_{i,j=1}^{m} b'_{ij}|x_i|\,|x_j| \\ &\leqslant \sum_{i,j=1}^{m} b'_{ij} x_i x_j = B'(v,v) \leqslant 0 \end{aligned}$$

Hence $B(|v|,|v|) = 0$ and this implies (Exercise!) that all the coefficients of $|v|$ are non-zero. Therefore $m = n$ and $b_{ij} = b'_{ij}$ for all $1 \leqslant i,j \leqslant n$, so that $m_{ij} = m'_{ij}$ and $\Gamma'_W = \Gamma_W$, hence a contradiction. ■

We also need the two following graphs:

Lemma 12.3

The graphs Z_4 $\bullet - \bullet \overset{5}{-} \bullet - \bullet$ and Z_5 $\bullet - \bullet - \bullet - \bullet \overset{5}{-} \bullet$ are neither positive definite nor positive semi-definite.

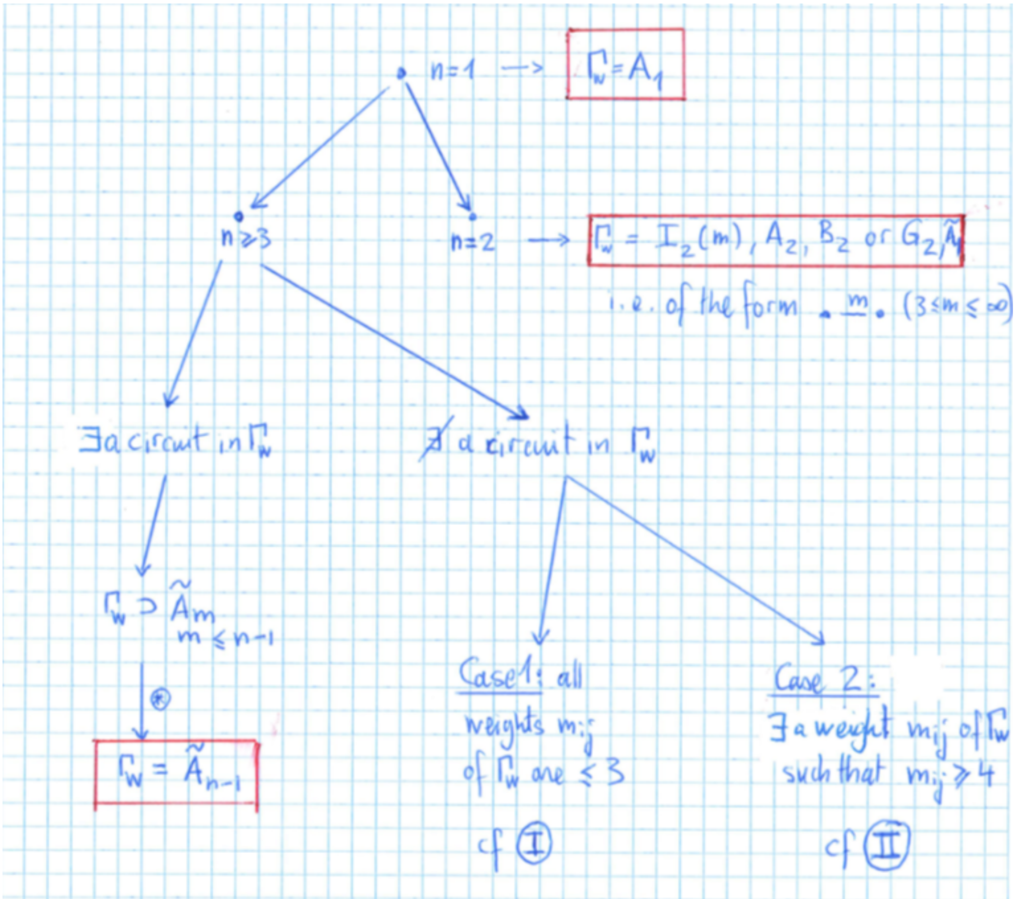
Proof: We find $\det(Z_4) = \frac{12-8\sqrt{5}}{64} < 0$ and $\det(Z_5) = \frac{2-\sqrt{5}}{16} < 0$. ■

Proof of Theorem A and Theorem B: necessary condition "⇐":
Let Γ_W be an irreducible Coxeter graph which is either positive definite or positive semi-definite. Let $n \in \mathbb{Z}_{>0}$ be the number of vertices of Γ_W. We have to prove that Γ_W belongs either to List A or to List B.
We use the following property:

Property ⊛ : Any proper subgraph of Γ_W is neither in List B, nor Z_4, nor Z_5.

Indeed, on the one hand, by Proposition 12.2 any proper subgraph of Γ' of Γ_W is positive definite, but on the other hand, we have seen in the proof of the sufficient condition ("⇐") of Theorem A and Theorem B and Lemma 12.3 that the graphs in List B$\sqcup\{Z_4, Z_5\}$ are <u>not</u> positive definite.

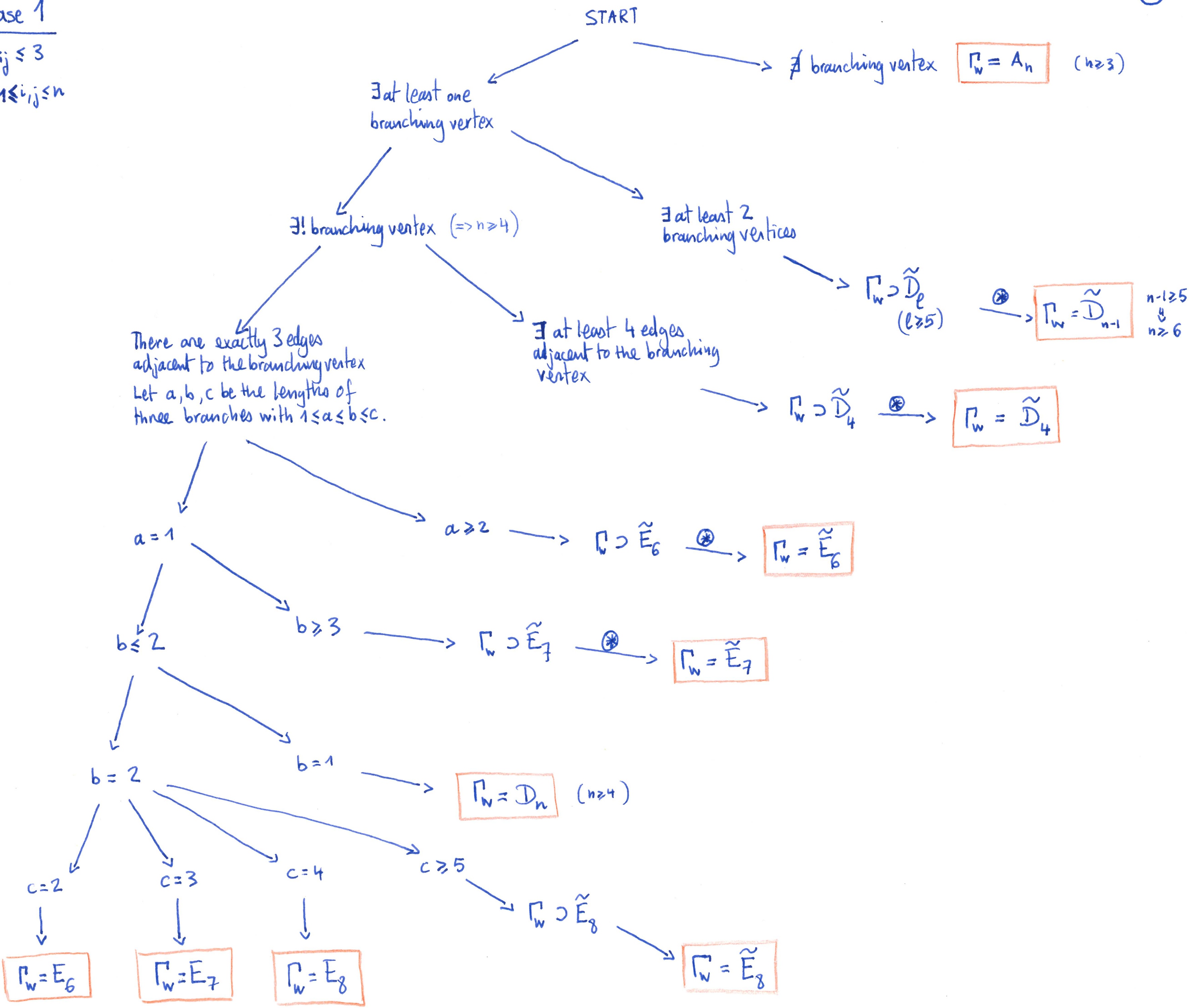
We are now going to prove that the fact that Γ_W has Property ⊛ implies that $\Gamma_W \in$ List A $\sqcup$ List B. We proceed in a constructive manner as follows:



∎

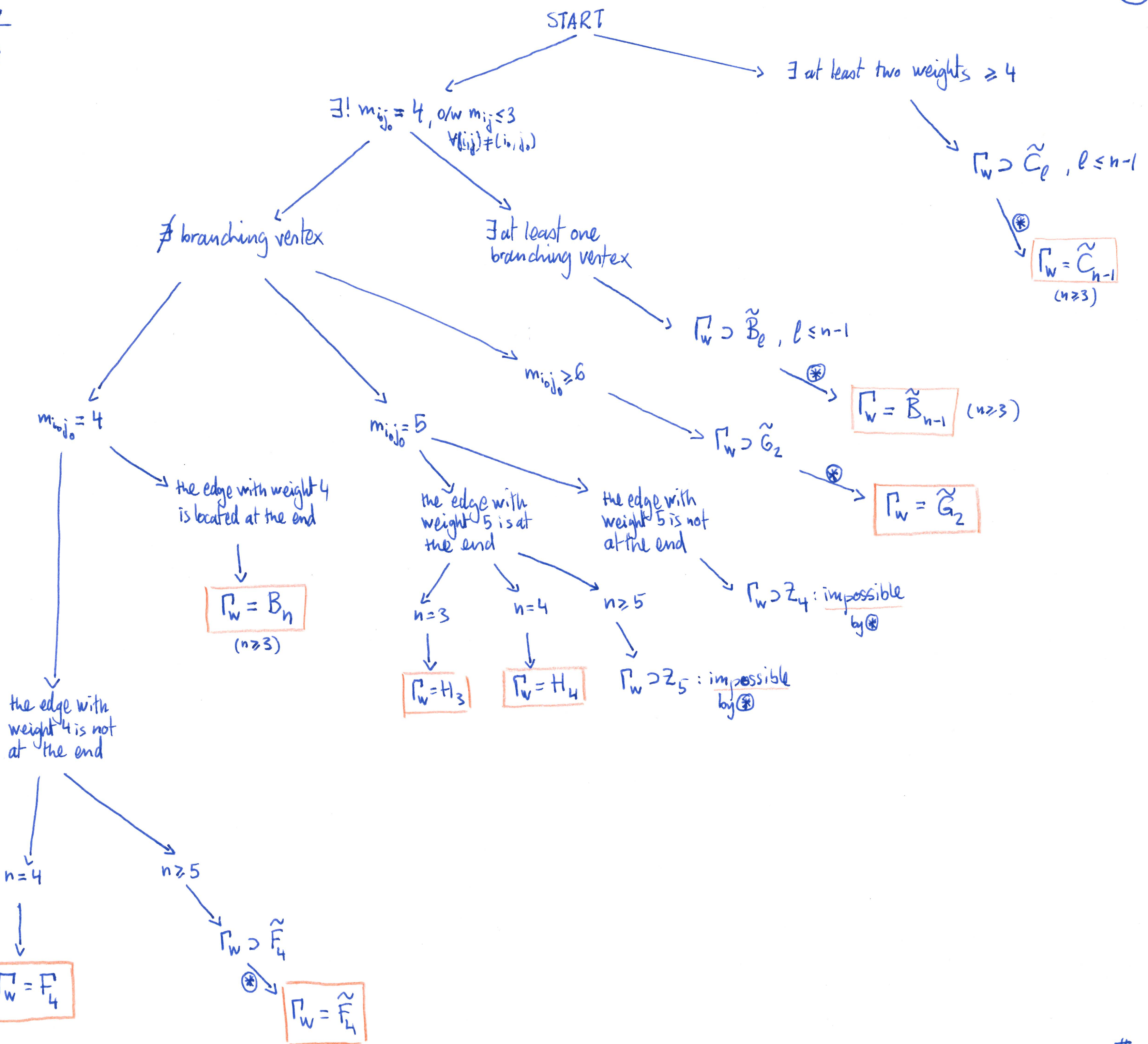
Case 1

$m_{ij} \leq 3$ $\forall\, 1 \leq i,j \leq n$



Case 2

$\exists\, m_{ij} \geq 4$



Appendix: Background Material: Group Theory

The aim of this chapter is to introduce formally two constructions of the theory of groups: *semi-direct products* and *presentations of groups*. Semi-direct products are useful when considering concrete groups, for instance in examples. Presentations describe groups by generators and relations in a concise way. They enable us to define Coxeter groups. Finally, in Section C, we present some well-known results of the representation theory of finite groups, which will enable us to classify the finite Coxeter groups.

References:

[Hum96] J. F. Humphreys, *A course in group theory*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1996.

[Joh90] D. L. Johnson, *Presentations of groups*, London Mathematical Society Student Texts, vol. 15, Cambridge University Press, Cambridge, 1990.

A Semi-direct products

The semi-direct product is a construction of the theory of groups, which allows us to build new groups from old ones. It is a natural generalisation of the direct product.

Definition A.1 (*Semi-direct product*)

A group G is said to be the (internal or inner) ***semi-direct product*** *of a normal subgroup $N \trianglelefteq G$ by a subgroup $H \leqslant G$ if the following conditions hold:*

(a) $G = NH$;

(b) $N \cap H = \{1\}$.

Notation: $G = N \rtimes H$.

Example 3

(1) *A direct product $G_1 \times G_2$ of two groups is the semi-direct product of $N := G_1 \times \{1\}$ by $H := \{1\} \times G_2$.*

(2) *$G = S_3$ is the semi-direct product of $N = C_3 = \langle(1\ 2\ 3)\rangle \trianglelefteq S_3$ and $H = C_2 = \langle(1\ 2)\rangle \leqslant S_3$. Hence $S_3 \cong C_3 \rtimes C_2$.*

Notice that, in particular, a semi-direct product of an abelian subgroup by an abelian subgroup need not be abelian.

(3) *More generally $G = S_n$ ($n \geqslant 3$) is a semi-direct product of $N = A_n \trianglelefteq S_n$ by $H = C_2 = \langle(1\ 2)\rangle$.*

Remark A.2

(a) *If G is a semi-direct product of N by H, then the 2nd Isomorphism Theorem yields*

$$G/N = HN/N \cong H/H \cap N = H/\{1\} \cong H$$

and this gives rise to a short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1\,.$$

Hence a semi-direct product of N by H is a special case of an extension of N by H.

(b) *In a semi-direct product $G = N \rtimes H$ of N by H, the subgroup H* acts by conjugation on *N, namely $\forall h \in H$,*

$$\begin{array}{rccc} \theta_h : & N & \longrightarrow & N \\ & n & \mapsto & hnh^{-1} \end{array}$$

is an automorphism of N. In addition $\theta_{hh'} = \theta_h \circ \theta_{h'}$ for every $h, h' \in H$, so that we have a group homomorphism

$$\begin{array}{rccc} \theta : & H & \longrightarrow & \mathrm{Aut}(N) \\ & h & \mapsto & \theta_h\,. \end{array}$$

Proposition A.3

With the above notation, N, H and θ are sufficient to reconstruct the group law on G.

Proof: Step 1. Each $g \in G$ can be written in a unique way as $g = nh$ where $n \in N$, $h \in H$: indeed by (a) and (b) of the Definition, if $g = nh = n'h'$ with $n, n' \in N$, $h, h' \in H$, then

$$n^{-1}n' = h(h')^{-1} \in N \cap H = \{1\}\,,$$

hence $n = n'$ and $h = h'$.

Step 2. Group law: Let $g_1 = n_1h_1, g_2 = n_2h_2 \in G$ with $n_1, n_2 \in N$, $h_1, h_2 \in H$ as above. Then

$$g_1g_2 = n_1h_1n_2h_2 = n_1 \underbrace{h_1n_2(h_1^{-1}}_{\theta_{h_1}(n_2)} h_1)h_2 = [n_1\theta_{h_1}(n_2)] \cdot [h_1h_2]\,.$$

■

With the construction of the group law in the latter proof in mind, we now consider the problem of constructing an "external" (or outer) semi-direct product of groups.

Proposition A.4

Let N and H be two arbitrary groups, and let $\theta : H \longrightarrow \mathrm{Aut}(N), h \mapsto \theta_h$ be a group homomorphism. Set $G := N \times H$ as a set. Then the binary operation

$$\begin{array}{rccl} \cdot : & G \times G & \longrightarrow & G \\ & \big((n_1, h_1), (n_2, h_2)\big) & \mapsto & (n_1, h_1) \cdot (n_2, h_2) := (n_1 \theta_{h_1}(n_2), h_1 h_2) \end{array}$$

defines a group law on G. The neutral element is $1_G = (1_N, 1_H)$ and the inverse of $(n, h) \in N \times H$ is $(n, h)^{-1} = (\theta_{h^{-1}}(n^{-1}), h^{-1})$.
Furthermore G is an internal semi-direct product of $N_0 := N \times \{1\} \cong N$ by $H_0 := \{1\} \times H \cong H$.

Proof: Exercise. ■

Definition A.5

*In the context of Proposition A.3 we say that G is the **external (or outer) semi-direct product** of N by H w.r.t. θ, and we write $G = N \rtimes_\theta H$.*

Example 4

Here are a few examples of very intuitive semi-direct products of groups, which you have very probably already encountered in other lectures, without knowing that they were semi-direct products:

(1) *If H acts trivially on N (i.e. $\theta_h = \mathrm{Id}_N\ \forall\, h \in H$), then $N \rtimes_\theta H = N \times H$.*

(2) *Let K be a field. Then*

$$\mathrm{GL}_n(K) = \mathrm{SL}_n(K) \rtimes \left\{\mathrm{diag}(\lambda, 1, \ldots, 1) \in \mathrm{GL}_n(K) \mid \lambda \in K^\times\right\} ,$$

where $\mathrm{diag}(\lambda, 1, \ldots, 1)$ is the diagonal matrix with (ordered) diagonal entries $\lambda, 1, \ldots, 1$.

(3) *Let K be a field and let*

$$B := \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in \mathrm{GL}_n(K) \right\} \quad (= \textit{upper triangular matrices}),$$

$$U := \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{GL}_n(K) \right\} \quad (= \textit{upper unitriangular matrices}),$$

$$T := \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \in \mathrm{GL}_n(K) \right\} \quad (= \textit{diagonal matrices}).$$

Clearly U is normal in B, since it is the kernel of the group homomorphism $B \longrightarrow T$ which sends a matrix in B to its diagonal. Thus B is a semi-direct product of U by T.

(4) *Let $C_m = \langle g \rangle$ and $C_n = \langle h \rangle$ ($m, n \in \mathbb{Z}_{\geqslant 1}$) be finite cyclic groups.*
Assume moreover that $k \in \mathbb{Z}$ is such that $k^n \equiv 1 \pmod m$ and set

$$\begin{array}{rccl} \theta : & C_n & \longrightarrow & \mathrm{Aut}(C_m) \\ & h^i & \mapsto & (\theta_h)^i , \end{array}$$

where $\theta_h : C_m \longrightarrow C_m, g \mapsto g^k$. Then

$$(\theta_h)^n(g) = (\theta_h)^{n-1}(g^k) = (\theta_h)^{n-2}(g^{k^2}) = \ldots = g^{k^n} = g$$

since $o(g) = m$ and $k^n \equiv 1 \pmod m$. Thus $(\theta_h)^n = \mathrm{Id}_{C_m}$ and θ is a group homomorphism. It follows that under these hypotheses there exists a semi-direct product of C_m by C_n w.r.t. to θ.

Particular case: $m \geqslant 1$, $n = 2$ and $k = -1$ yield the dihedral group D_{2m} of order $2m$ with generators g (of order m) and h (of order 2) and the relation $\theta_h(g) = hgh^{-1} = g^{-1}$.

B Presentations of groups

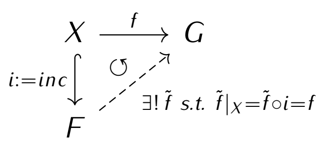
Idea: describe a group using a set of generators and a set of relations between these generators.

Examples:	(1)	$C_m = \langle g \rangle = \langle g \mid g^m = 1 \rangle$	1 generator: g 1 relation: $g^m = 1$
	(2)	$D_{2m} = C_m \rtimes_\theta C_2$	2 generators: g, h 3 relations: $g^m = 1, h^2 = 1, hgh^{-1} = g^{-1}$
	(3)	$\mathbb{Z} = \langle 1_\mathbb{Z} \rangle$	1 generator: $1_\mathbb{Z}$ no relation (⇝ "free group")

To begin with we examine free groups and generators.

Definition B.1 (*Free group / Universal property of free groups*)

*Let X be a set. A **free group with basis** X (or **free group on** X) is a group F containing X as a subset and satisfying the following universal property: For any group G and for any (set-theoretic) map $f : X \longrightarrow G$, there exists a unique group homomorphism $\tilde{f} : F \longrightarrow G$ such that $\tilde{f}|_X = f$, or in other words such that the following diagram commutes:*

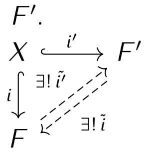


*Moreover, $|X|$ is called the **rank** of F.*

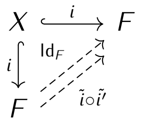
Proposition B.2

If F exists, then F is the unique free group with basis X up to a unique isomorphism.

Proof: Assume F' is another free group with basis X.
Let $i : X \hookrightarrow F$ be the canonical inclusion of X in F and let $i' : X \hookrightarrow F'$ be the canonical inclusion of X in F'.



By the universal property of Definition B.1, there exists:
- a unique group homomorphism $\tilde{i'} : F \longrightarrow F'$ s.t. $i' = \tilde{i'} \circ i$; and
- a unique group homomorphism $\tilde{i} : F' \longrightarrow F$ s.t. $i = \tilde{i} \circ i'$.



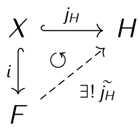
Then $(\tilde{i}\circ\tilde{i}')|_X = i$, but obviously we also have $\mathrm{Id}_F|_X = i$. Therefore, by uniqueness, we have $\tilde{i}\circ\tilde{i}' = \mathrm{Id}_F$.

A similar argument yields $\tilde{i}'\circ\tilde{i} = \mathrm{Id}_{F'}$, hence F and F' are isomorphic, up to a unique isomorphism, namely $\tilde{i}$ with inverse $\tilde{i}'$. ■

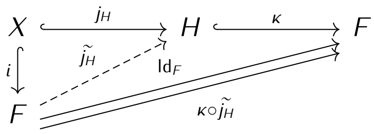
Proposition B.3

If F is a free group with basis X, then X generates F.

Proof: Let $H := \langle X\rangle$ be the subgroup of F generated by X, and let $j_H := X \hookrightarrow H$ denote the canonical inclusion of X in H. By the universal property of Definition B.1, there exists a unique group homomorphism $\widetilde{j_H}$ such that $\widetilde{j_H}\circ i = j_H$:



Therefore, letting $\kappa : H \hookrightarrow F$ denote the canonical inclusion of H in F, we have the following commutative diagram:



Thus by uniqueness $\kappa\circ\widetilde{j_H} = \mathrm{Id}_F$, implying that $\widetilde{j_H} : H \longrightarrow F$ is injective. Thus

$$F = \mathrm{Im}(\mathrm{Id}_F) = \mathrm{Im}(\kappa\circ\widetilde{j_H}) = \mathrm{Im}(\widetilde{j_H}) \subseteq H$$

and it follows that $F = H$. The claim follows. ■

Theorem B.4

For any set X, there exists a free group F with basis X.

Proof: Set $X := \{x_\alpha \mid \alpha\in I\}$ where I is a set in bijection with X, set $Y := \{y_\alpha \mid \alpha\in I\}$ in bijection with X but disjoint from X, i.e. $X\cap Y = \varnothing$, and let $Z := X\cup Y$.
Furthermore, set $E := \bigcup_{n=0}^{\infty} Z^n$, where $Z^0 := \{(\ )\}$ (i.e. a singleton), $Z^1 := Z$, $Z^2 := Z\times Z$, ...
Then E becomes a monoid for the concatenation of sequences, that is

$$\underbrace{(z_1,\dots,z_n)}_{\in Z^n}\cdot\underbrace{(z_1',\dots,z_m')}_{\in Z^m} := \underbrace{(z_1,\dots,z_n,z_1',\dots,z_n')}_{\in Z^{n+m}} .$$

The law $\cdot$ is clearly associative by definition, and the neutral element is the empty sequence $(\ )\in Z^0$.
Define the following *Elementary Operations* on the elements of E:

Type (1):	add in a sequence $(z_1,\dots,z_n)$ two consecutive elements x_α, y_α and obtain $(z_1,\dots,z_k,x_\alpha,y_\alpha,z_{k+1},\dots,z_n)$
Type (1bis):	add in a sequence $(z_1,\dots,z_n)$ two consecutive elements y_α, x_α and obtain $(z_1,\dots,z_m,y_\alpha,x_\alpha,z_{m+1},\dots,z_n)$
Type (2):	remove from a sequence $(z_1,\dots,z_n)$ two consecutive elements x_α, y_α and obtain $(z_1,\dots,z_r,\check{x}_\alpha,\check{y}_\alpha,z_{r+1},\dots,z_n)$
Type (2bis):	remove from a sequence $(z_1,\dots,z_n)$ two consecutive elements y_α, x_α and obtain $(z_1,\dots,z_s,\check{y}_\alpha,\check{x}_\alpha,z_{s+1},\dots,z_n)$

Now define an equivalence relation $\sim$ on E as follows:

two sequences in E are equivalent $:\Longleftrightarrow$ the 2nd sequence can be obtain from the 1st sequence through a succession of Elementary Operations of type (1), (1bis), (2) and (2bis).

It is indeed easily checked that this relation is:
– reflexive: simply use an empty sequence of Elementary Operations;
– symmetric: since each Elementary Operation is invertible;
– transitive: since 2 consecutive sequences of Elementary Operations is again a sequence of Elementary Operations.

Now set $F := E/\sim$, and write $[z_1, \dots, z_n]$ for the equivalence class of $(z_1, \dots, z_n)$ in $F = E/\sim$.

<u>Claim 1:</u> <u>The above monoid law on E induces a monoid law on F</u>.
The induced law on F is: $[z_1, \dots, z_n] \cdot [z'_1, \dots, z'_m] = [z_1, \dots, z_n, z'_1, \dots, z'_m]$.
It is well-defined: if $(z_1, \dots, z_n) \sim (t_1, \dots, t_k)$ and $(z'_1, \dots, z'_m) \sim (t'_1, \dots, t'_l)$, then

$$\begin{aligned}(z_1, \dots, z_n) \cdot (z'_1, \dots, z'_m) &= (z_1, \dots, z_n, z'_1, \dots, z'_m) \\ &\sim (t_1, \dots, t_k, z'_1, \dots, z'_m) \quad \text{via Elementary Operations on the 1st part} \\ &\sim (t_1, \dots, t_k, t'_1, \dots, t'_l) \quad \text{via Elementary Operations on the 2nd part} \\ &= (t_1, \dots, t_n) \cdot (t'_1, \dots, t'_m)\end{aligned}$$

The associativity is clear, and the neutral element is $[(\ )]$. The claim follows.

<u>Claim 2:</u> <u>F endowed with the monoid law defined in Claim 1 is a group</u>.
Inverses: the inverse of $[z_1, \dots, z_n] \in F$ is the equivalence of the sequence class obtained from $(z_1, \dots, z_n)$ by reversing the order and replacing each x_α with y_α and each y_α with x_α. (Obvious by definition of $\sim$.)

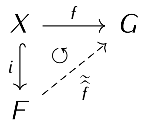
<u>Claim 3:</u> <u>F is a free group on X</u>.
Let G be a group and $f : X \longrightarrow G$ be a map. Define

$$\begin{array}{cccc}\widehat{f}: & E & \longrightarrow & G \\ & (z_1, \dots, z_n) & \mapsto & f(z_1) \cdot \ \cdots \ \cdot f(z_n)\,,\end{array}$$

where f is defined on Y by $f(y_\alpha) := f(x_\alpha^{-1})$ for every $y_\alpha \in Y$.
Thus, if $(z_1, \dots, z_n) \sim (t_1, \dots, t_k)$, then $\widehat{f}(z_1, \dots, z_n) = \widehat{f}(t_1, \dots, t_k)$ by definition of f on Y. Hence f induces a map

$$\begin{array}{cccc}\widehat{\widehat{f}}: & F & \longrightarrow & G \\ & [z_1, \dots, z_n] & \mapsto & f(z_1) \cdot \ \cdots \ \cdot f(z_n)\,,\end{array}$$

By construction $\widehat{f}$ is a monoid homomorphism, therfore so is $\widehat{\widehat{f}}$, but since F and G are groups, $\widehat{\widehat{f}}$ is in fact a group homomorphism. Hence we have a commutative diagram



where $i : X \longrightarrow F, x \mapsto [x]$ is the canonical inclusion.

Finally, notice that the definition of $\widehat{\widehat{f}}$ is forced if we want $\widehat{\widehat{f}}$ to be a group homorphism, hence we have uniqueness of $\widehat{\widehat{f}}$, and the universal property of Definition B.1 is satisfied. ■

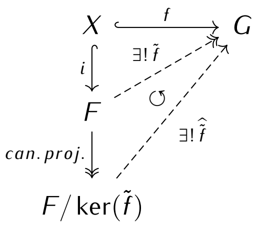
Notation and Terminology

- To lighten notation, we identify $[x_\alpha] \in F$ with x_α, hence $[y_\alpha]$ with x_α^{-1}, and $[z_1, \dots, z_n]$ with $z_1 \cdots z_n$ in F.
- A sequence $(z_1, \dots, z_n) \in E$ with each letter z_i $(1 \leqslant i \leqslant n)$ equal to an element $x_{\alpha_i} \in X$ or $x_{\alpha_i}^{-1}$ is called a **word** in the generators $\{x_\alpha \mid \alpha \in I\}$. Each word defines an element of F via: $(z_1, \dots, z_n) \mapsto z_1 \cdots z_n \in F$. By abuse of language, we then often also call $z_1 \cdots z_n \in F$ a *word*.
- Two words are called **equivalent** $:\Longleftrightarrow$ they define the same element of F.
- If $(z_1, \dots, z_n) \in Z_n \subseteq E$ $(n \in \mathbb{Z}_{\geqslant 0})$, then n is called the **length** of the word $(z_1, \dots, z_n)$.
- A word is said to be **reduced** if it has minimal length amongst all the words which are equivalent to this word.

Proposition B.5

Every group G is isomorphic to a factor group of a free group.

Proof: Let $S := \{g_\alpha \in G \mid \alpha \in I\}$ be a set of generators for G (in the worst case, take $I = G$). Let $X := \{x_\alpha \mid \alpha \in I\}$ be a set in bijection with S, and let F be the free group on X. Let $i : X \hookrightarrow F$ denote the canonical inclusion.



By the universal property of free groups the map $f : X \hookrightarrow G, x_\alpha \mapsto g_\alpha$ induces a unique group homomorphism $\tilde{f} : F \longrightarrow G$ such that $\tilde{f} \circ i = f$. Clearly $\tilde{f}$ is surjective since the generators of G are all $\operatorname{Im}(\tilde{f})$. Therefore the 1st Isomorphism Theorem yields $G \cong F/\ker(\tilde{f})$.

■

We can now consider relations between the generators of groups:

Notation and Terminology

Let $S := \{g_\alpha \in G \mid \alpha \in I\}$ be a set of generators for the group G, let $X := \{x_\alpha \mid \alpha \in I\}$ be in bijection with S, and let F be the free group on X.

By the previous proof, $G \cong F/N$, where $N := \ker(\tilde{f})$ $(g_\alpha \leftrightarrow \overline{x_\alpha} = x_\alpha N$ via the homomorphism $\hat{\tilde{f}}$).

Any word $(z_1, \dots, z_n)$ in the x_α's which defines an element of F in N is mapped in G to an expression of the form

$$\overline{z_1} \cdots \overline{z_n} = 1_G\ , \qquad \text{where } \overline{z_i} := \text{image of } z_i \text{ in } G \text{ under the canonical homomorphism.}$$

In this case, the word $(z_1, \dots, z_n)$ is called a **relation in the group G for the set of generators S**.

Now let $R := \{r_\beta \mid \beta \in J\}$ be a set of generators of N as normal subgroup of F (this means that N is generated by the set of all conjugates of R). Such a set R is called a **set of defining relations of G**.

Then the ordered pair (X, R) is called a **presentation** of G, and we write

$$G = \langle X \mid R \rangle = \langle \{x_\alpha\}_{\alpha \in I} \mid \{r_\beta\}_{\beta \in J} \rangle .$$

The group G is said to be **finitely presented** if it admits a presentation $G = \langle X \mid R\rangle$, where both $|X|, |R| < \infty$. In this case, by abuse of notation, we also often write presentations under the form

$$G = \langle x_1, \ldots, x_{|X|} \mid r_1 = 1, \ldots, r_{|R|} = 1\rangle .$$

Example 5

The cyclic group $C_n = \{1, g, \ldots, g^{n-1}\}$ of order $n \in \mathbb{Z}_{\geqslant 1}$ generated by $S := \{g\}$. In this case, we have:

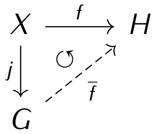
$X = \{x\}$
$R = \{x^n\}$
$F = \langle x\rangle \cong (C_\infty, \cdot)$
$C_\infty \xrightarrow{\tilde{f}} C_n, x \mapsto g$ *has a kernel generated by x^n as a normal subgroup. Then $C_n = \langle\{x\} \mid \{x^n\}\rangle$.*

By abuse of notation, we write simply $C_n = \langle x \mid x^n\rangle$ or also $C_n = \langle x \mid x^n = 1\rangle$.

Proposition B.6 (*Universal property of presentations*)

Let G be a group generated by $S = \{s_\alpha \mid \alpha \in I\}$, isomorphic to a quotient of a free group F on $X = \{x_\alpha \mid \alpha \in I\}$ in bijection with S. Let $R := \{r_\beta \mid \beta \in J\}$ be a set of relations in G.

Then G admits the presentation $G = \langle X \mid R\rangle$ if and only if G satisfies the following universal property:



For every group H, and for every set-theoretic map $f : X \longrightarrow H$ such that $\tilde{f}(r_\beta) = 1_H \ \forall\ r_\beta \in R$, there exists a unique group homomorphism $\overline{f} : G \longrightarrow H$ such that $\overline{f} \circ j = f$, where $j : X \longrightarrow G, x_\alpha \mapsto s_\alpha$, and $\tilde{f}$ is the unique extension of f to the free group F on X.

Proof: "$\Rightarrow$": Suppose that $G = \langle X \mid R\rangle$. Therefore $G \cong F/N$, where N is generated by R as normal subgroup. Thus the condition $\tilde{f}(r_\beta) = 1_H \ \forall\ r_\beta \in R$ implies that $N \subseteq \ker(\tilde{f})$, since

$$\tilde{f}(zr_\beta z^{-1}) = \tilde{f}(z)\underbrace{\tilde{f}(r_\beta)}_{=1_H}\tilde{f}(z)^{-1} = 1_H \qquad \forall\, r_\beta \in R, \forall\, z \in F.$$

Therefore, by the universal property of the quotient, $\tilde{f}$ induces a unique group homomorphism $\overline{f} : G \cong F/N \longrightarrow H$ such that $\overline{f} \circ \pi = \tilde{f}$, where $\pi : F \longrightarrow F/N$ is the canonical epimorphism. Now, if $i : X \longrightarrow F$ denotes the canonical inclusion, then $j = \pi \circ i$, and as a consequence we have $\overline{f} \circ j = f$.

"$\Leftarrow$": Conversely, assume that G satisfies the universal property of the statement (i.e. relatively to X, F, R). Set $N := \overline{R}$ for the normal closure of R. Then we have two group homomorphisms:

$$\begin{array}{rccc} \varphi\colon & F/N & \longrightarrow & G \\ & \overline{x_\alpha} & \mapsto & s_\alpha \end{array}$$

induced by $\tilde{f} : F \longrightarrow G$, and

$$\begin{array}{rccc} \psi\colon & G & \longrightarrow & F/N \\ & s_\alpha & \mapsto & \overline{x_\alpha} \end{array}$$

given by the universal property. Then clearly $\varphi \circ \psi(s_\alpha) = \varphi(\overline{x_\alpha}) = s_\alpha$ for each $\alpha \in I$, so that $\varphi \circ \psi = \mathrm{Id}_G$ and similarly $\psi \circ \varphi = \mathrm{Id}_{F/R}$. The claim follows. ■

Example 6

Consider the finite dihedral group D_{2m} of order $2m$ with $2 \leqslant m < \infty$. We can assume that D_{2m} is generated by

$$r := \text{ rotation of angle } \frac{2\pi}{m} \text{ and } s := \text{ symmetry through the origin in } \mathbb{R}^2 .$$

Then $\langle r \rangle \cong C_m \subseteq G$, $\langle s \rangle \cong C_2$ and we have seen that $D_{2m} = \langle r \rangle \rtimes \langle s \rangle$ with three obvious relations $r^m = 1$, $s^2 = 1$, and $srs^{-1} = r^{-1}$.

***Claim**: D_{2m} admits the presentation $\langle r, s \mid r^m = 1, s^2 = 1, srs^{-1} = r^{-1} \rangle$.*

In order to prove the Claim, we let F be the free group on $X := \{x, y\}$, $R := \{x^m, y^2, yxy^{-1}x\}$, $N \trianglelefteq F$ be the normal subgroup generated by R, and $G := F/N$ so that

$$G = \langle \overline{x}, \overline{y} \mid \overline{x}^m = 1, \overline{y}^2 = 1, \overline{y}\,\overline{x}\,\overline{y}^{-1}\overline{x} = 1 \rangle .$$

By the universal property of presentations the map

$$\begin{array}{rccc} f: & \{x, y\} & \longrightarrow & D_{2m} \\ & x & \mapsto & r \\ & y & \mapsto & s \end{array}$$

induces a group homomorphism

$$\begin{array}{rccc} \overline{f}: & G & \longrightarrow & D_{2m} \\ & \overline{x} & \mapsto & r \\ & \overline{y} & \mapsto & s , \end{array}$$

which is clearly surjective since $D_{2m} = \langle r, s \rangle$. In order to prove that $\overline{f}$ is injective, we prove that G is a group of order at most $2m$. Recall that each element of G is an expression in $\overline{x}, \overline{y}, \overline{x}^{-1}, \overline{y}^{-1}$, hence actually an expression in $\overline{x}, \overline{y}$, since $\overline{x}^{-1} = \overline{x}^{m-1}$ and $\overline{y}^{-1} = \overline{y}$. Moreover, $\overline{yxy}^{-1} = \overline{x}^{-1}$ implies $\overline{yx} = \overline{x}^{-1}\overline{y}$, hence we are left with expressions of the form

$$\overline{x}^a \overline{y}^b \qquad \text{with } 0 \leqslant a \leqslant m-1 \text{ and } 0 \leqslant b \leqslant 1 .$$

Thus we have $|G| \leqslant 2m$, and it follows that $\overline{f}$ is an isomorphism.

Notice that if we remove the relation $r^m = 1$, we can also formally define an infinite dihedral group D_∞ via the following presentation

$$D_\infty := \langle r, s \mid s^2 = 1, srs^{-1} = r^{-1} \rangle .$$

Theorem B.7

Let G be a group generated by two distinct elements, s and t, both of order 2. Then $G \cong D_{2m}$, where $2 \leqslant m \leqslant \infty$. Moreover, m is the order of st in G, and

$$G = \langle s, t \mid s^2 = 1, t^2 = 1, (st)^m = 1 \rangle .$$

($m = \infty$ simply means "no relation".)

Proof: Set $r := st$ and let m be the order of r.

Firstly, note that $m \geqslant 2$, since $m = 1 \Rightarrow st = 1 \Rightarrow s = t^{-1} = t$ as $t^2 = 1$. Secondly, we have the relation $srs^{-1} = r^{-1}$, since

$$srs^{-1} = \underbrace{s(s}_{=1_G} t)s^{-1} = ts^{-1} = t^{-1}s^{-1} = (st)^{-1} = r^{-1}\,.$$

Clearly G can be generated by r and s as $r = st$ and so $t = sr$.
Now, $H := \langle r\rangle \cong C_m$ and $H \trianglelefteq G$ since

$$srs^{-1} = r^{-1} \in H \quad \text{and} \quad rrr^{-1} = r \in H \quad (\text{or because } |G : H| = 2)\,.$$

Set $C := \langle s\rangle \cong C_2$.

Claim: $s \notin H$.

Indeed, assuming $s \in H$ yields $s = r^i = (st)^i$ for some $0 \leqslant i \leqslant m-1$. Hence

$$1 = s^2 = s(st)^i = (ts)^{i-1}t = \underbrace{(ts\cdots t)}_{\text{length } i-1}\, s\, \underbrace{(ts\cdots t)}_{\text{length } i-1}\,,$$

so that conjugating by t, then s, then ..., then t, we get $1 = s$, contradicting the assumption that $o(s) = 2$. The claim follows.

Therefore, we have proved that $G = HC$ and $H \cap C = \{1\}$, so that $G = H \rtimes C = D_{2m}$ as seen in the previous section.

Finally, to prove that G admits the presentation $\langle s, t \mid s^2 = 1, t^2 = 1, (st)^m = 1\rangle$, we apply the universal property of presentations twice to the maps

$$\begin{array}{rccl} f: & \{x_s, x_r\} & \longrightarrow & \langle s, t \mid s^2 = 1, t^2 = 1, (st)^m = 1\rangle \\ & x_s & \mapsto & s \\ & x_r & \mapsto & st \end{array}$$

and

$$\begin{array}{rccl} g: & \{y_s, y_t\} & \longrightarrow & G = \langle r, s \mid r^m = 1, s^2 = 1, srs^{-1} = 1\rangle \\ & y_s & \mapsto & s \\ & y_t & \mapsto & sr\ . \end{array}$$

This yields the existence of two group homomorphisms

$$\overline{f} : G = \langle r, s \mid r^m = 1, s^2 = 1, srs^{-1} = 1\rangle \longrightarrow \langle s, t \mid s^2 = 1, t^2 = 1, (st)^m = 1\rangle$$

and

$$\overline{g} : \langle s, t \mid s^2 = 1, t^2 = 1, (st)^m = 1\rangle \longrightarrow G = \langle r, s \mid r^m = 1, s^2 = 1, srs^{-1} = 1\rangle$$

such that $\overline{g}\overline{f} = \mathrm{Id}$ and $\overline{f}\overline{g} = \mathrm{Id}$. (Here you should check the details for yourself!) ■

C Representation theory and $\mathbb{R}$-bilinear forms

We assume throughout this section that W is an arbitrary group, $\sigma : W \longrightarrow \mathrm{GL}(V)$ an arbitrary representation of W over a finite-dimensional $\mathbb{R}$-vector space V (i.e. a group homomorphism from W to $\mathrm{GL}(V)$) and we consider its dual representation σ^* defined by

$$\begin{array}{rccl} \sigma^*: & W & \longrightarrow & \mathrm{GL}(V^*) \\ & w & \mapsto & \sigma^*(w) := {}^t(\sigma(w^{-1}))\,. \end{array}$$

Given $w \in W$ and $v \in V$ we set $w.v := \sigma(w)(v)$ and given $w \in W$ and $f \in V^*$, we set $w.f := \sigma^*(w)(f)$.

We present here some standard results of representation theory, which we partially accept without proof. We need the following terminology:

Definition C.1

(a) *A subspace $U \subseteq V$ is called* ***W-invariant*** *if $w.U \subseteq U$ ($\Leftrightarrow w.U = U$) for every $w \in W$.*

(b) *An* ***endomorphism*** *of σ is an $\mathbb{R}$-linear map $\varphi : V \longrightarrow V$ such that $\varphi(w.v) = w.\varphi(v)$ (i.e. $\varphi(\sigma(w)(v)) = \sigma(w)(\varphi(v))$) for every $w \in W$ and $v \in V$.*

(c) *The representation σ is called* ***irreducible*** *if V has exactly two distinct W-invariant subspaces, namely $\{0\}$ and V itself.*

(d) *The representation σ is called* ***absolutely irreducible*** *if σ is irreducible and any endomorphism φ of σ has the form $\varphi = \lambda \cdot \mathrm{Id}_V$ for some $\lambda \in \mathbb{R}$.*

(e) *An $\mathbb{R}$-bilinear form $B : V \times V \longrightarrow \mathbb{R}$ is called* ***W-invariant*** *if*

$$B(w.v, w.v') = B(v, v') \qquad \forall\, w \in W,\ \forall\, v, v' \in V\,.$$

Maschke's Theorem (over $\mathbb{R}$)

Assume W is a finite group and let $\sigma : W \longrightarrow \mathrm{GL}(V)$ be is a representation of W. If $U \subseteq V$ is a W-invariant subspace, then there exists a W-invariant subspace $U' \subseteq V$ such that $W = U \oplus U'$.

Proof: Omitted. ■

Proposition C.2

Let $\sigma : W \longrightarrow \mathrm{GL}(V)$ be an absolutely irreducible representation of W and let $B : V \times V \longrightarrow \mathbb{R}$ be a non-zero W-invariant $\mathbb{R}$-bilinear form. Then:

(a) *B is non-degenerate;*

(b) *any W-invariant $\mathbb{R}$-bilinear form $B' : V \times V \longrightarrow \mathbb{R}$ is a scalar multiple of B.*

Proof: Set $\widehat{B} : V \longrightarrow V^*, u \mapsto B(-, u)$.

Claim 1: B is W-invariant $\iff$ $\widehat{B}$ is a so-called **homomorphism of representations between σ and σ^***, in other words such that $\widehat{B}(w.v) = w.\widehat{B}(v)\ \forall\, w \in W,\ \forall\, v \in V$.

Proof of Claim 1: Exercise!

(a) It follows from Claim 1 that $\ker \widehat{B}$ is W-invariant, because for every $w \in W$ we have:

$$u \in \ker \widehat{B} \quad \Rightarrow \quad \widehat{B}(w.u) = w.\underbrace{\widehat{B}(u)}_{=0} = 0 \quad \Rightarrow \quad w.u \in \ker \widehat{B}.$$

Now, as B is non-zero, $\ker \widehat{B} \neq V$, hence the only possibility remaining is $\ker \widehat{B} = \{0\}$ because we assume that σ is irreducible. It follows that $\widehat{B}$ is injective, and hence bijective, because $\dim_{\mathbb{R}} V < \infty$ implies that $\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} V^*$. Therefore, B is non-degenerate.

(b) Let $B' : V \times V \longrightarrow \mathbb{R}$ be a second W-invariant $\mathbb{R}$-bilinear form. Since B is non-degenerate by (a), $\widehat{B} : V \longrightarrow V^*$ is an isomorphism. Therefore, there exists an $\mathbb{R}$-linear map $\varphi : V \longrightarrow V$ such that

$$B'(v', v) = B(v', \varphi(v)) \qquad \forall\, v', v \in V\,.$$

Concretely, one may take $\varphi = \widehat{B}^{-1} \circ \widehat{B'}$, since $B'(-, v) = \widehat{B'}(v) = \widehat{B} \circ \varphi(v) = B(-, \varphi(v))$.
Now, since B and B' are W-invariant, by Claim 1, both $\widehat{B}$ and $\widehat{B'}$ are homomorphisms between σ and σ^*, therefore so is $\varphi = \widehat{B}^{-1} \circ \widehat{B'}$.
Furthermore, σ being absolutely irreducible, there exists $\lambda \in \mathbb{R}$ such that $\varphi = \lambda \cdot \mathrm{Id}_V$. Hence

$$B'(v', v) = B(v', \varphi(v)) = B(v', \lambda \cdot v) = \lambda \cdot B(v', v)$$

for every $v', v \in V$ and it follows that $B' = \lambda \cdot B$. ■

Index of Notation

General symbols

$\mathbb{C}$	field of complex numbers
Id_M	identity map on the set M
$\mathrm{Im}(f)$	image of the map f
$\ker(\varphi)$	kernel of the morphism φ
$\mathbb{N}$	the natural numbers without 0
$\mathbb{N}_0$	the natural numbers with 0
$\mathbb{P}$	the prime numbers in $\mathbb{Z}$
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{Z}$	ring of integer numbers
$\mathbb{Z}_{\geqslant a}, \mathbb{Z}_{>a}, \mathbb{Z}_{\leqslant a}, \mathbb{Z}_{<a}$	$\{m \in \mathbb{Z} \mid m \geqslant a$ (resp. $m > a, m \geqslant a, m < a)\}$
$\|X\|$	cardinality of the set X
δ_{ij}	Kronecker's delta
$\bigcup$	union
$\coprod$	disjoint union
$\bigcap$	intersection
$\sum$	summation symbol
$\prod$, $\times$	cartesian/direct product
$\rtimes$	semi-direct product
$\oplus$	direct sum
$\varnothing$	empty set
$\forall$	for all
$\exists$	there exists
$\cong$	isomorphism
$f\|_S$	restriction of the map f to the subset S
$\hookrightarrow$	injective map
$\twoheadrightarrow$	surjective map

Algebra

$\mathrm{Aut}(G)$	automorphism group of the group G
$\mathfrak{A}_n$	alternating group on n letters
C_m	cyclic group of order m in multiplicative notation
$C_G(x)$	centraliser of the element x in G
$C_G(H)$	centraliser of the subgroup H in G
D_{2n}	dihedral group of order $2n$
$\det$	determinant
$\mathrm{End}(A)$	endomorphism ring of the abelian group A
G/N	quotient group G modulo N
$\mathrm{GL}_n(K)$	general linear group over K
$H \leqslant G$, $H < G$	H is a subgroup of G, resp. a proper subgroup
$N \trianglelefteq G$	N is a normal subgroup G
$N_G(H)$	normaliser of H in G
$N \rtimes_\theta H$	semi-direct product of N in H w.r.t. θ
$\mathrm{PGL}_n(K)$	projective linear group over K
$\mathfrak{S}_n$	symmetric group on n letters
$\mathrm{SL}_n(K)$	special linear group over K
V^*	$\mathbb{R}$-dual of the $\mathbb{R}$-vector space V
$\mathbb{Z}/m\mathbb{Z}$	cyclic group of order m in additive notation
${}^t\varphi$	transpose of the linear map/matrix φ
${}^xg$	conjugate of the group element g by x, i.e. gxg^{-1}
$\langle g \rangle \subseteq G$	subgroup of G generated by g
$G = \langle X \mid R \rangle$	presentation for the group G
$\lvert G : H \rvert$	index of the subgroup H in G
$\overline{x} \in G/N$	class of $x \in G$ in the quotient group G/N
$\{1\}$, 1	trivial group